

N-BARRIER MAXIMUM PRINCIPLE FOR DEGENERATE ELLIPTIC SYSTEMS AND ITS APPLICATION

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ABSTRACT. In this paper, we prove the N-barrier maximum principle, which extends the result in [5] from linear diffusion equations to nonlinear diffusion equations, for a wide class of degenerate elliptic systems of porous medium type. The N-barrier maximum principle provides a priori upper and lower bounds of the solutions to the above-mentioned degenerate nonlinear diffusion equations including the Shigesada-Kawasaki-Teramoto model as a special case. As an application of the N-barrier maximum principle to a coexistence problem in ecology, we show the nonexistence of waves in a three-species degenerate elliptic systems.

1. INTRODUCTION AND MAIN RESULTS

The main perspective of the paper is to establish the *N-barrier maximum principle* (NBMP, see [5, 7]) for degenerate elliptic systems. To be more precise, we study

$$d_i (u_i^m)_{xx} + \theta (u_i)_x + u_i^{l_i} f_i(u_1, u_2, \dots, u_n) = 0, \quad x \in \mathbb{R}, \quad i = 1, 2, \dots, n, \quad (1.1)$$

where $u_i = u_i(x)$, $d_i, l_i > 0$, $\theta \in \mathbb{R}$, and $f_i(u_1, u_2, \dots, u_n) \in C^0(\mathbb{R}^+ \times \mathbb{R}^+ \times \dots \times \mathbb{R}^+)$ for $i = 1, 2, \dots, n$. The NBMP for the linear diffusion case $m = 1$ has been presented in [5, 7]. In this sequel we will deal with the nonlinear diffusion case $m > 1$ based on the N-barrier method developed in [5, 7].

We couple (1.1) with the prescribed Dirichlet conditions at $x = \pm\infty$:

$$(u_1, u_2, \dots, u_n)(-\infty) = \mathbf{e}_-, \quad (u_1, u_2, \dots, u_n)(\infty) = \mathbf{e}_+, \quad (1.2)$$

where

$$\mathbf{e}_-, \mathbf{e}_+ \in \left\{ (u_1, u_2, \dots, u_n) \mid u_i^{l_i} f_i(u_1, u_2, \dots, u_n) = 0 \ (i = 1, 2, \dots, n), u_1, u_2, \dots, u_n \geq 0 \right\} \quad (1.3)$$

are the equilibria of (1.1) which connect the solution $(u_1, u_2, \dots, u_n)(x)$ at $x = -\infty$ and $x = \infty$. This leads to the boundary value problem of (1.1) and (1.2):

$$(\text{BVP}) \begin{cases} d_i (u_i^m)_{xx} + \theta (u_i)_x + u_i^{l_i} f_i(u_1, u_2, \dots, u_n) = 0, & x \in \mathbb{R}, \quad i = 1, 2, \dots, n, \\ (u_1, u_2, \dots, u_n)(-\infty) = \mathbf{e}_-, & (u_1, u_2, \dots, u_n)(\infty) = \mathbf{e}_+. \end{cases}$$

Throughout, we assume, unless otherwise stated, that the following hypothesis on $f_i(u_1, u_2, \dots, u_n)$ is satisfied:

[H] For $i = 1, 2, \dots, n$, there exist $\bar{u}_i > \underline{u}_i > 0$ such that

$$f_i(u_1, u_2, \dots, u_n) \geq 0 \quad \text{whenever } (u_1, u_2, \dots, u_n) \in \mathcal{R};$$

$$f_i(u_1, u_2, \dots, u_n) \leq 0 \quad \text{whenever } (u_1, u_2, \dots, u_n) \in \bar{\mathcal{R}},$$

where

$$\begin{aligned} \mathcal{R} &= \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \frac{u_i}{\underline{u}_i} \leq 1, u_1, u_2, \dots, u_n > 0 \right\}; \\ \bar{\mathcal{R}} &= \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \frac{u_i}{\bar{u}_i} \geq 1, u_1, u_2, \dots, u_n \geq 0 \right\}. \end{aligned}$$

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Also, we denote by χ the characteristic function:

$$\chi = \begin{cases} 0, & \text{if } \mathbf{e}_+ = (0, \dots, 0) \text{ or } \mathbf{e}_- = (0, \dots, 0), \\ 1, & \text{otherwise.} \end{cases} \quad (1.4)$$

The main contribution of the N-barrier maximum principle is that it provides rather generic a priori upper and lower bounds for the linear combination of the components of a vector-valued solution which hold for a wide class of reaction terms and boundary conditions. In particular, the key ingredient in the proof relies on the delicate construction of an appropriate N-barrier which allows us to establish the a priori estimates by contradiction.

Theorem 1.1 (NBMP for $m = 1$, [5, 7]). *Assume that [H] holds. Given any set of $\alpha_i > 0$ ($i = 1, 2, \dots, n$), suppose that $(u_1(x), u_2(x), \dots, u_n(x))$ is a nonnegative C^2 solution to (BVP) with $m = 1$. Then*

$$\lambda \leq \sum_{i=1}^n \alpha_i u_i(x) \leq \bar{\lambda}, \quad x \in \mathbb{R}, \quad (1.5)$$

where

$$\bar{\lambda} = \frac{\left(\max_{1 \leq i \leq n} \alpha_i \bar{u}_i \right) \left(\max_{1 \leq i \leq n} d_i \right)}{\min_{1 \leq i \leq n} d_i}, \quad (1.6)$$

$$\lambda = \frac{\left(\min_{1 \leq i \leq n} \alpha_i u_i \right) \left(\min_{1 \leq i \leq n} d_i \right)}{\max_{1 \leq i \leq n} d_i} \chi, \quad (1.7)$$

with χ given by (1.4).

(BVP) arises from the study of traveling waves in the Shigesada-Kawasaki-Teramoto (SKT) model

$$(\text{SKT}) \begin{cases} u_t = \Delta(u(d_1 + \rho_{11}u + \rho_{12}v)) + u(\sigma_1 - c_{11}u - c_{12}v), & y \in \Omega, \quad t > 0, \\ v_t = \Delta(v(d_2 + \rho_{21}u + \rho_{22}v)) + v(\sigma_2 - c_{21}u - c_{22}v), & y \in \Omega, \quad t > 0, \end{cases}$$

which was proposed by Shigesada, Kawasaki and Teramoto ([36]) in 1979 to study the spatial segregation problem for two competing species. Here $u(y, t)$ and $v(y, t)$ stand for the density of the two species u and v , respectively, and $\Omega \subseteq \mathbb{R}^n$ is the habitat of the two species. $d_1 \Delta u$ and $d_2 \Delta v$ come from the random movements of individual species with diffusion rates $d_1, d_2 > 0$. Meanwhile, the terms $\Delta(u(\rho_{11}u + \rho_{12}v))$ and $\Delta(v(\rho_{21}u + \rho_{22}v))$ include the *self-diffusion* and *cross-diffusion* due to the directed movements of the individuals toward favorable habitats. The coefficients ρ_{11} and ρ_{22} are referred to as the self-diffusion rates, while ρ_{12} and ρ_{21} are the cross-diffusion rates. In addition, the coefficients σ_i, c_{ii} ($i = 1, 2$), and c_{ij} ($i, j = 1, 2$ with $i \neq j$) are the intrinsic growth rates, the intra-specific competition rates, and the inter-specific competition rates, which are all assumed to be positive, respectively.

To tackle the problem as to which species will survive in a competitive system is of importance in ecology. To this end, we consider traveling wave solutions, which are solutions of the form

$$(u(y, t), v(y, t)) = (u(x), v(x)), \quad x = y - \theta t, \quad (1.8)$$

where $x \in \mathbb{R}$ and $\theta \in \mathbb{R}$ is the propagation speed of the traveling wave. Ecologically, the sign of θ indicates which species is stronger and can survive. Inserting (1.8) into (SKT) with $\Omega = \mathbb{R}$ leads to

$$(\text{SKT-tw}) \begin{cases} (u(d_1 + \rho_{11}u + \rho_{12}v))_{xx} + \theta u_x + u(\sigma_1 - c_{11}u - c_{12}v) = 0, & x \in \mathbb{R}, \\ (v(d_2 + \rho_{21}u + \rho_{22}v))_{xx} + \theta v_x + v(\sigma_2 - c_{21}u - c_{22}v) = 0, & x \in \mathbb{R}. \end{cases}$$

When the self-diffusion and the cross-diffusion effects are neglected or $\rho_{11} = \rho_{12} = \rho_{21} = \rho_{22} = 0$, **(SKT)** with $\Omega = \mathbb{R}$ and **(SKT-tw)** reduce respectively to

$$(\mathbf{LV}) \begin{cases} u_t = d_1 \Delta u + u(\sigma_1 - c_{11}u - c_{12}v), & y \in \mathbb{R}, \quad t > 0, \\ v_t = d_2 \Delta v + v(\sigma_2 - c_{21}u - c_{22}v), & y \in \mathbb{R}, \quad t > 0, \end{cases}$$

and

$$(\mathbf{LV-tw}) \begin{cases} d_1 u_{xx} + \theta u_x + u(\sigma_1 - c_{11}u - c_{12}v) = 0, & x \in \mathbb{R}, \\ d_2 v_{xx} + \theta v_x + v(\sigma_2 - c_{21}u - c_{22}v) = 0, & x \in \mathbb{R}, \end{cases}$$

where **(LV)** is the celebrated Lotka-Volterra competition-diffusion system of two species and the NBMP for **(LV-tw)** has been established by applying Theorem 1.1 for **(LV-tw)** ([5]).

We illustrate our motivation for establishing Theorem 1.1 for **(LV-tw)** as follows. When the habitat of the two competing species u and v is resource-limited, the investigation of the total mass or the total density of the two species u and v is essential. This gives rise to the problem of estimating the total density $u(x) + v(x)$ in **(LV-tw)**. In addition, another issue which motivates us to study the estimate of $u(x) + v(x)$ is the measurement of the *species evenness index* \mathcal{J} for **(LV-tw)**. \mathcal{J} is defined via *Shannon's diversity index* \mathcal{H} ([3, 11, 30, 37]), i.e.

$$\mathcal{J} = \frac{\mathcal{H}}{\ln(s)}, \quad (1.9)$$

where

$$\mathcal{H} = - \sum_{i=1}^s \iota_i \cdot \ln(\iota_i), \quad (1.10)$$

s is the total number of species, and ι_i is the proportion of the i -th species determined by dividing the number of the i -th species species by the total number of all species. The species evenness index \mathcal{J} for **(LV-tw)** is given by

$$\mathcal{J} = - \frac{1}{(\ln 2)(u+v)} \left(u \ln \left(\frac{u}{u+v} \right) + v \ln \left(\frac{v}{u+v} \right) \right). \quad (1.11)$$

We see $u(x) + v(x)$ is involved in the calculation of \mathcal{J} .

Another problem we are concerned with is the parameter dependence on the estimate of $u(x) + v(x)$. When $d_1 = d_2$, upper and lower bounds of $u(x) + v(x)$ are given in [6] by an approach based on the elliptic maximum principle. For the case of $d_1 \neq d_2$, an affirmative answer to an even more general problem of estimating $\alpha u + \beta v$, where $\alpha, \beta > 0$ are arbitrary constants, is given by means of Theorem 1.1.

On the there hand, we are led to (1.1) with $m = n = 2$ and $l_i = 1$ ($i = 1, 2, \dots, n$) when $d_1 = d_2 = \rho_{12} = \rho_{21} = 0$ in **(SKT-tw)**. We therefore, address the following problem.

Q: Under **[H]**, establish the NBMP for **(BVP)**, i.e. find nontrivial lower and upper bounds (depending on the coefficients in **(BVP)**) of $\sum_{i=1}^n \alpha_i u_i(x)$, where $\alpha_i > 0$ ($i = 1, 2, \dots, n$) are arbitrary positive constants.

Our main result is that **(BVP)** enjoys the following N-barrier maximum principle, which gives an affirmative answer to **Q**. Indeed, we have

Theorem 1.2 (NBMP for (BVP)). Assume that **[H]** holds. Given any set of $\alpha_i > 0$ ($i = 1, 2, \dots, n$), suppose that $(u_1(x), u_2(x), \dots, u_n(x))$ is a nonnegative C^2 solution to **(BVP)** with $m > 1$. Then

$$\lambda \leq \sum_{i=1}^n \alpha_i u_i(x) \leq \bar{\lambda}, \quad x \in \mathbb{R}, \quad (1.12)$$

where

$$\bar{\lambda} = \sqrt[m]{\left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt[m]{d_i}} \right)^{2(m-1)} \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}} \right) \left(\max_{1 \leq i \leq n} \alpha_i d_i \bar{u}_i^m \right)}, \quad (1.13)$$

$$\lambda = \sqrt[m]{\left(\sum_{i=1}^n \frac{1}{m^{-1}\sqrt{\alpha_i d_i u_i^m}}\right)^{1-m} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt{d_i}}\right)^{1-m} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i}\right)^2} \chi, \quad (1.14)$$

with χ given by (1.4).

We note that except the case in which either $\mathbf{e}_+ = (0, \dots, 0)$ or $\mathbf{e}_- = (0, \dots, 0)$, the boundary conditions at $\pm\infty$ do not play any role in determining the upper and lower bounds given by Theorem 1.2. For either $\mathbf{e}_+ = (0, \dots, 0)$ or $\mathbf{e}_- = (0, \dots, 0)$, we clearly have only the trivial lower bound zero.

To illustrate Theorem 1.2, we present an example. Suppose that $m = n = 2$, $l_i = 1$ and $f_i(u_1, u_2) = u_i(\sigma_i - c_{i1}u_1 - c_{i2}u_2)$ ($i = 1, 2$). Then **(BVP)** becomes

$$\textbf{(NDC-tw)} \begin{cases} d_1(u_1^2)_{xx} + \theta(u_1)_x + u_1(\sigma_1 - c_{11}u_1 - c_{12}u_2) = 0, & x \in \mathbb{R}, \\ d_2(u_2^2)_{xx} + \theta(u_2)_x + u_2(\sigma_2 - c_{21}u_1 - c_{22}u_2) = 0, & x \in \mathbb{R}, \\ (u_1, u_2)(-\infty) = \mathbf{e}_-, \quad (u_1, u_2)(+\infty) = \mathbf{e}_+, \end{cases}$$

where

$$\mathbf{e}_-, \mathbf{e}_+ \in \left\{ (u_1, u_2) \mid u_i(\sigma_i - c_{i1}u_1 - c_{i2}u_2) = 0 \ (i = 1, 2), u_1, u_2 \geq 0 \right\}. \quad (1.15)$$

The degenerate elliptic system **(NDC-tw)** arises from the study of traveling waves in **(SKT)** without the presence of diffusion and cross-diffusion, and Ω replaced by \mathbb{R} , i.e.

$$\textbf{(NDC)} \begin{cases} (u_1)_t = d_1(u_1^2)_{yy} + u_1(\sigma_1 - c_{11}u_1 - c_{12}u_2), & y \in \mathbb{R}, \ t > 0, \\ (u_2)_t = d_2(u_2^2)_{yy} + u_2(\sigma_2 - c_{21}u_1 - c_{22}u_2), & y \in \mathbb{R}, \ t > 0. \end{cases}$$

The nonlinear diffusion-competition system **(NDC)** has been studied, for example in [13]. Under suitable restrictions on the coefficients, explicit spatially periodic stationary solutions to **(NDC)** can be found. In addition, for appropriate diffusion coefficients the existence of an explicit, unbounded traveling wave to **(NDC)** is proved under either strong or weak competition. An immediate consequence of Theorem 1.2 is the following NBMP for **(NDC-tw)**.

Corollary 1.3 (NBMP for NDC-tw). *Assume that $(u(x), v(x))$ is a nonnegative C^2 solution to **(NDC-tw)**. For any set of $\alpha_i > 0$ ($i = 1, 2$), we have*

$$\lambda \leq \alpha_1 u_1(x) + \alpha_2 u_2(x) \leq \bar{\lambda}, \quad x \in \mathbb{R}, \quad (1.16)$$

where

$$\bar{\lambda} = \left(\frac{\alpha_1}{d_1} + \frac{\alpha_2}{d_2} \right) \sqrt{\max\left(\frac{d_1}{\alpha_1}, \frac{d_2}{\alpha_2}\right) \max\left(\alpha_1 d_1 \bar{u}_1^2, \alpha_2 d_2 \bar{u}_2^2\right)}, \quad (1.17)$$

$$\lambda = d_1 d_2 u_1 u_2 \min\left(\frac{\alpha_1}{d_1}, \frac{\alpha_2}{d_2}\right) \sqrt{\frac{\alpha_1 \alpha_2}{(\alpha_1 d_1 u_1^2 + \alpha_2 d_2 u_2^2)(\alpha_1 d_2 + \alpha_2 d_1)}} \chi, \quad (1.18)$$

with χ given by (1.4) and

$$\begin{aligned} \bar{u}_1 &= \max\left(\frac{\sigma_1}{c_{11}}, \frac{\sigma_2}{c_{21}}\right), & \bar{u}_2 &= \max\left(\frac{\sigma_1}{c_{12}}, \frac{\sigma_2}{c_{22}}\right), \\ \underline{u}_1 &= \min\left(\frac{\sigma_1}{c_{11}}, \frac{\sigma_2}{c_{21}}\right), & \underline{u}_2 &= \min\left(\frac{\sigma_1}{c_{12}}, \frac{\sigma_2}{c_{22}}\right), \end{aligned} \quad (1.19)$$

Proof. We apply Theorem 1.2 to prove Corollary 1.3. Due to (1.19), it can be verified that **[H]** is satisfied. Indeed, we have

$$\mathcal{R} = \left\{ (u_1, u_2) \mid \sum_{i=1}^2 \frac{u_i}{\min_{j=1,2} \frac{\sigma_j}{c_{ji}}} \leq 1, \ u_1, u_2 \geq 0 \right\};$$

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$$\bar{\mathcal{R}} = \left\{ (u_1, u_2) \left| \sum_{i=1}^2 \frac{u_i}{\max_{j=1,2} \frac{\sigma_j}{c_{ji}}} \geq 1, u_1, u_2 \geq 0 \right. \right\}.$$

Since $\min_{j=1,2} \frac{\sigma_j}{c_{ji}} \left(\max_{j=1,2} \frac{\sigma_j}{c_{ji}} \right)$, respectively) is the smallest (largest, respectively) u_i -intercept of the two planes $\sigma_i - c_{i1} u_1 - c_{i2} u_2 = 0$ ($i = 1, 2$), we see that

$$\sigma_i - c_{i1} u_1 - c_{i2} u_2 \geq 0 \quad \text{whenever } (u_1, u_2) \in \mathcal{R};$$

$$\sigma_i - c_{i1} u_1 - c_{i2} u_2 \leq 0 \quad \text{whenever } (u_1, u_2) \in \bar{\mathcal{R}},$$

for each $i = 1, 2$. The desired result follows from Theorem 1.2. □

As an interesting application of the linear diffusion NBMP (Theorem 1.1), we investigate the situation where one exotic competing species (say, w) invades the ecological system of two native species (say, u and v) that are competing in the absence of w . A problem related to *competitive exclusion* ([2, 18, 19, 21, 25, 38]) or *competitor-mediated coexistence* ([4, 22, 26]) then arises. The Lotka-Volterra system of three competing species is usually used to model this situation ([1, 10, 12, 16, 17, 23, 24, 26, 29, 9, 40]). Under this situation, the traveling wave solution $(u(x), v(x), w(x))$ satisfies the following system:

$$\begin{cases} d_1 u_{xx} + \theta u_x + u(\sigma_1 - c_{11} u - c_{12} v - c_{13} w) = 0, & x \in \mathbb{R}, \\ d_2 v_{xx} + \theta v_x + v(\sigma_2 - c_{21} u - c_{22} v - c_{23} w) = 0, & x \in \mathbb{R}, \\ d_3 w_{xx} + \theta w_x + w(\sigma_3 - c_{31} u - c_{32} v - c_{33} w) = 0, & x \in \mathbb{R}. \end{cases} \quad (1.20)$$

Clearly, (1.1) includes (1.20) as a special case. For (1.20), existence of solutions with profiles of one-hump waves coupled with the boundary conditions

$$(u, v, w)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, 0 \right), \quad (u, v, w)(\infty) = \left(0, \frac{\sigma_2}{c_{22}}, 0 \right). \quad (1.21)$$

is investigated under certain assumptions on the parameters by finding exact solutions ([8, 6]) and using the numerical tracking method AUTO ([8]). A one-hump wave is referred to as a traveling wave consisting of a forward front v , a backward front u , and a pulse w in the middle. On the other hand, nonexistence of solutions for the problem (1.20), (1.21) is established by means of the NBMP (Theorem 1.1) as well as the elliptic maximum principle under certain conditions ([6, 5]).

Recently, new dynamical patterns exhibited by the solutions of the Lotka-Volterra system of three competing species have been found in [26], where traveling wave solutions of the three species (i.e. solutions of (1.20)) are used as building blocks (1.20) to generate dynamical patterns in which three species coexist. This numerical evidence demonstrates (indicates) from the viewpoint of dynamical coexistence of the three species the great importance of the one-hump waves in the problem (1.20), (1.21).

The linear diffusion terms in (1.20) are based on Fick's law in which the population flux is proportional to the gradient of the population density. In some situations, however, evidences from field studies have shown the inadequacy of this model. Due to population pressure, the phenomenon that species tend to avoid crowded can be characterized by the population flux which depends on both the population density and its gradient ([27, 35, 39]). Gurney and Nisbet considered the nonlinear diffusion effect described above, and proposed the following the model ([14, 15])

$$u_t = (u u_x)_x + u(u - 1), \quad (1.22)$$

where the population flux is proportional to u and u_x . Based on porous medium version of the Fisher equation (1.22) ([28, 34, 33]), (1.20) becomes

$$\begin{cases} d_1 (u^2)_{xx} + \theta u_x + u(\sigma_1 - c_{11} u - c_{12} v - c_{13} w) = 0, & x \in \mathbb{R}, \\ d_2 (v^2)_{xx} + \theta v_x + v(\sigma_2 - c_{21} u - c_{22} v - c_{23} w) = 0, & x \in \mathbb{R}, \\ d_3 (w^2)_{xx} + \theta w_x + w(\sigma_3 - c_{31} u - c_{32} v - c_{33} w) = 0, & x \in \mathbb{R}, \end{cases} \quad (1.23)$$

For the existence of solutions of the problem (1.23), (1.21), it seems as far as we know, not available in the literature. As a starting point to study this problem, we instead find the conditions on the parameters under which the solutions do not exist. With the aid of the NBMP for the problem (1.23), (1.21), this can be achieved as the following nonexistence result shows.

Theorem 1.4 (Nonexistence of three species waves). *Under either (i) or (ii), (1.23) admits no positive solution $(u(x), v(x), w(x))$ with $u(x), v(x), w(x) \not\equiv \text{constant}$.*

(i) Let $\phi_1 = \sigma_1 - c_{13} \sigma_3 c_{33}^{-1}$ and $\phi_2 = \sigma_2 - c_{23} \sigma_3 c_{33}^{-1}$. Assume that the following hypotheses hold:

[H0] $(u, v)(\pm\infty) \neq (0, 0)$;

[H1] $\max_{x \in \mathbb{R}} w(x) = w(x_0)$ for some $x_0 \in \mathbb{R}$;

[H2] $\phi_1, \phi_2 > 0$;

[H3] $\lambda_* := d_1 d_2 u_* v_* \min\left(\frac{c_{31}}{d_1}, \frac{c_{32}}{d_2}\right) \sqrt{\frac{c_{31} c_{32}}{(c_{31} d_1 u_*^2 + c_{32} d_2 v_*^2)(c_{31} d_2 + c_{32} d_1)}} \geq \sigma_3$, where

$$u_* = \min\left(\frac{\phi_1}{c_{11}}, \frac{\phi_2}{c_{21}}\right), \quad v_* = \min\left(\frac{\phi_1}{c_{12}}, \frac{\phi_2}{c_{22}}\right). \quad (1.24)$$

(ii) Assume that the following hypotheses hold:

[H4] $\min_{x \in \mathbb{R}} w(x) = w(x_0)$ for some $x_0 \in \mathbb{R}$;

[H5] $\lambda^* := \left(\frac{c_{31}}{d_1} + \frac{c_{32}}{d_2}\right) \sqrt{\max\left(\frac{d_1}{c_{31}}, \frac{d_2}{c_{32}}\right) \max\left(c_{31} d_1 \bar{u}^{*2}, c_{32} d_2 \bar{v}^{*2}\right)} < \sigma_3$, where

$$\bar{u}^* = \max\left(\frac{\sigma_1}{c_{11}}, \frac{\sigma_2}{c_{21}}\right), \quad \bar{v}^* = \max\left(\frac{\sigma_1}{c_{12}}, \frac{\sigma_2}{c_{22}}\right). \quad (1.25)$$

[H6] $w(\pm\infty) := w_{\pm\infty}$, where either $w_{-\infty} < \frac{1}{c_{33}}(\sigma_3 - \lambda^*)$ or $w_{+\infty} < \frac{1}{c_{33}}(\sigma_3 - \lambda^*)$.

We note that when the boundary conditions are imposed at $x = \pm\infty$ like (1.21), hypotheses [H0] and [H1] are simultaneously satisfied. Roughly speaking, (i) of Theorem 1.4 says from the viewpoint of ecology that when the intrinsic growth rate σ_3 of w is sufficiently small (i.e. [H3]), the three species u , v and w cannot coexist in the ecological system modeled by (1.23), (1.21). In other words, competitor-mediated coexistence cannot occur in such a circumstance. On the other hand, [H6] is satisfied when the boundary conditions are

$$(u, v, w)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0, 0\right), \quad (u, v, w)(\infty) = (0, \tilde{v}, \tilde{w}), \quad (1.26)$$

where $v = \tilde{v}$, $w = \tilde{w}$ solves

$$\sigma_2 - c_{22} v - c_{23} w = 0, \quad \sigma_3 - c_{32} v - c_{33} w = 0 \quad (1.27)$$

or

$$\tilde{v} = \frac{c_{23} \sigma_3 - c_{33} \sigma_2}{c_{23} c_{32} - c_{22} c_{33}}, \quad \tilde{w} = \frac{c_{32} \sigma_2 - c_{22} \sigma_3}{c_{23} c_{32} - c_{22} c_{33}}, \quad (1.28)$$

whenever the coexistence state (\tilde{v}, \tilde{w}) exists. [H4] is an extra hypothesis on the profile of the wave. As a consequence, (ii) of Theorem 1.4 asserts that under certain conditions on the boundary conditions (i.e. [H6]) and on the profile of the wave (i.e. [H4]), coexistence among the three species u , v and w cannot occur when the intrinsic growth rate σ_3 of w is sufficiently large (i.e. [H5]).

The remainder of this paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.2. As an application of Theorem 1.2, we show in Section 3 the nonexistence result of three species in Theorem 1.4.

In Section 4, we propose some open problems concerning the NBMP. Finally, some exact traveling wave solutions and the solutions of a system of algebraic equations needed in the proof of Theorem 1.2 are given in the Appendix (Section 5).

2. PROOF OF THEOREM 1.2

Proposition 1 (Lower bound in NBMP). *Suppose that $u_i(x) \in C^2(\mathbb{R})$ with $u_i(x) \geq 0$ ($i = 1, 2, \dots, n$) and satisfy the following differential inequalities and asymptotic behavior:*

$$(BVP-u) \begin{cases} d_i (u_i^m)_{xx} + \theta (u_i)_x + u_i^{l_i} f_i(u_1, u_2, \dots, u_n) \leq 0, & x \in \mathbb{R}, \quad i = 1, 2, \dots, n, \\ (u_1, u_2, \dots, u_n)(-\infty) = \mathbf{e}_-, & (u_1, u_2, \dots, u_n)(\infty) = \mathbf{e}_+, \end{cases}$$

where \mathbf{e}_- and \mathbf{e}_+ are given by (1.3). If the hypothesis

[H] For $i = 1, 2, \dots, n$, there exist $\underline{u}_i > 0$ such that

$$f_i(u_1, u_2, \dots, u_n) \geq 0 \quad \text{whenever } (u_1, u_2, \dots, u_n) \in \mathcal{R},$$

where \mathcal{R} is as defined in [H]

holds, then we have for any $\alpha_i > 0$ ($i = 1, 2, \dots, n$)

$$\sum_{i=1}^n \alpha_i u_i(x) \geq \sqrt[m]{\left(\sum_{i=1}^n \frac{1}{m^{-1} \sqrt[m]{\alpha_i d_i u_i^m}} \right)^{1-m} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1} \sqrt[m]{d_i}} \right)^{1-m} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i} \right)^2} \chi \quad (2.1)$$

where χ is defined as in (1.4).

Proof. For the case where $\mathbf{e}_+ = (0, \dots, 0)$ or $\mathbf{e}_- = (0, \dots, 0)$, a trivial lower bound of $\sum_{i=1}^n \alpha_i u_i(x)$ is 0. It suffices to show (2.1) for the case $\mathbf{e}_+ \neq (0, \dots, 0)$ and $\mathbf{e}_- \neq (0, \dots, 0)$. To this end, we let

$$p(x) = \sum_{i=1}^n \alpha_i u_i(x); \quad (2.2)$$

$$q(x) = \sum_{i=1}^n \alpha_i d_i u_i^m(x). \quad (2.3)$$

Adding the n equations in (BVP-u), we obtain a single equation involving $p(x)$ and $q(x)$

$$\frac{d^2 q(x)}{dx^2} + \theta \frac{dp(x)}{dx} + F(u_1(x), u_2(x), \dots, u_n(x)) \leq 0, \quad x \in \mathbb{R}, \quad (2.4)$$

where $F(u_1, u_2, \dots, u_n) := \sum_{i=1}^n \alpha_i u_i^{l_i} f_i(u_1, u_2, \dots, u_n)$. First of all, we show how to construct the N -barrier.

Determining an appropriate N -barrier is crucial in establishing (2.1). The construction of the N -barrier consists of determining the positive parameters $\lambda_1, \lambda_2, \eta_1$ and η_2 such that the two hyper-ellipsoids $\sum_{i=1}^n \alpha_i d_i u_i^m =$

λ_1 and $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$, and the two hyperplanes $\sum_{i=1}^n \alpha_i u_i = \eta_1$ and $\sum_{i=1}^n \alpha_i u_i = \eta_2$ satisfy the relationship

$$\mathcal{P}_{\eta_2} \subset \mathcal{Q}_{\lambda_2} \subset \mathcal{P}_{\eta_1} \subset \mathcal{Q}_{\lambda_1} \subset \mathcal{R}, \quad (2.5)$$

where

$$\mathcal{P}_{\eta} = \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \alpha_i u_i \leq \eta, \quad u_1, u_2, \dots, u_n \geq 0 \right\}; \quad (2.6)$$

$$\mathcal{Q}_{\lambda} = \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \alpha_i d_i u_i^m \leq \lambda, \quad u_1, u_2, \dots, u_n \geq 0 \right\}. \quad (2.7)$$

The hyper-ellipsoids $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ and $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$, and the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ form the N-barrier; it turns out that the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_2$ determines a lower bound of $p(x)$. We follow the three steps below to construct the N-barrier:

- (1) Let the hyperplane $\sum_{i=1}^n \frac{u_i}{u_i} = 1$ be tangent to the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ at (u_1, u_2, \dots, u_n) with $u_1, u_2, \dots, u_n > 0$ such that $\mathcal{Q}_{\lambda_1} \subset \mathcal{R}$. This leads to the following equations:

$$\alpha_i d_i u_i^{m-1} u_i = \alpha_j d_j u_j^{m-1} u_j, \quad i, j = 1, 2, \dots, n; \quad (2.8)$$

$$\sum_{i=1}^n \frac{u_i}{u_i} = 1; \quad (2.9)$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1. \quad (2.10)$$

By Lemma 5.1 (see Section 5), λ_1 is determined by

$$\lambda_1 = \left(\sum_{i=1}^n \frac{1}{m-1 \sqrt{\alpha_i d_i u_i^m}} \right)^{1-m}. \quad (2.11)$$

- (2) Setting

$$\eta_1 = \sqrt[m]{\lambda_1 \min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i}}, \quad (2.12)$$

the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ has the n intercepts $\left(\frac{\eta_1}{\alpha_1}, 0, \dots, 0\right), \left(0, \frac{\eta_1}{\alpha_2}, 0, \dots, 0\right), \dots$, and $\left(0, 0, \dots, 0, \frac{\eta_1}{\alpha_n}\right)$ and the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ has the n intercepts $\left(\sqrt[m]{\frac{\lambda_1}{\alpha_1 d_1}}, 0, \dots, 0\right), \left(0, \sqrt[m]{\frac{\lambda_1}{\alpha_2 d_2}}, 0, \dots, 0\right), \dots$, and $\left(0, 0, \dots, 0, \sqrt[m]{\frac{\lambda_1}{\alpha_n d_n}}\right)$. It is easy to verify that $\mathcal{P}_{\eta_1} \subset \mathcal{Q}_{\lambda_1}$ since $\frac{\eta_1}{\alpha_j} \leq \sqrt[m]{\frac{\lambda_1}{\alpha_j d_j}}$ for $j = 1, 2, \dots, n$. Indeed, we have

$$\begin{aligned} \frac{\eta_1}{\alpha_j} &= \left(\min_{1 \leq i \leq n} \frac{\lambda_1 \alpha_i^{m-1}}{d_i} \right)^{\frac{1}{m}} \frac{1}{\alpha_j} \\ &\leq \left(\frac{\lambda_1 \alpha_j^{m-1}}{d_j \alpha_j^m} \right)^{\frac{1}{m}} = \left(\frac{\lambda_1}{\alpha_j d_j} \right)^{\frac{1}{m}}. \end{aligned} \quad (2.13)$$

- (3) Let the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ be tangent to the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$ at (u_1, u_2, \dots, u_n) with $u_1, u_2, \dots, u_n > 0$ such that $\mathcal{Q}_{\lambda_2} \subset \mathcal{P}_{\eta_1}$. This leads to the following equations:

$$d_i u_i^{m-1} = d_j u_j^{m-1}, \quad i, j = 1, 2, \dots, n; \quad (2.14)$$

$$\sum_{i=1}^n \alpha_i u_i = \eta_1; \quad (2.15)$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2. \quad (2.16)$$

Employing Lemma 5.2 in Section 5, we obtain

$$\lambda_2 = \eta_1^m \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt{d_i}} \right)^{1-m}. \quad (2.17)$$

Steps (i)~(iii) complete the construction of the N-barrier. As in step (ii), we determine η_2 by

$$\eta_2 = \sqrt[m]{\lambda_2 \min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i}} \quad (2.18)$$

such that $\mathcal{P}_{\eta_2} \subset \mathcal{Q}_{\lambda_2}$. From (2.11), (2.12), (2.17) and (2.18), it follows immediately that η_2 is given by

$$\begin{aligned} \eta_2 &= \lambda_2^{\frac{1}{m}} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i} \right)^{\frac{1}{m}} = \eta_1 \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt{d_i}} \right)^{\frac{1-m}{m}} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i} \right)^{\frac{1}{m}} \\ &= \lambda_1^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt{d_i}} \right)^{\frac{1-m}{m}} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i} \right)^{\frac{2}{m}} \\ &= \left(\sum_{i=1}^n \frac{1}{m^{-1}\sqrt{\alpha_i d_i u_i^m}} \right)^{\frac{1-m}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1}\sqrt{d_i}} \right)^{\frac{1-m}{m}} \left(\min_{1 \leq i \leq n} \frac{\alpha_i^{m-1}}{d_i} \right)^{\frac{2}{m}}. \end{aligned} \quad (2.19)$$

(The construction of the N-barrier for the simplified case $m = n = 2$ is illustrated in Remark 2.1, which provides an intuitive idea of the construction of the N-barrier in higher dimensional cases.)

We claim that $q(x) \geq \lambda_2$, $x \in \mathbb{R}$. This proves (2.1), i.e $q(x) \geq \eta_2$, $x \in \mathbb{R}$ since the $\alpha_i > 0$ ($i = 1, 2, \dots, n$) are arbitrary and the relationship $\mathcal{P}_{\eta_2} \subset \mathcal{Q}_{\lambda_2}$ holds. Now we prove the claim by contradiction. Suppose that, contrary to our claim, there exists $z \in \mathbb{R}$ such that $q(z) < \lambda_2$. Since $u_i(x) \in C^2(\mathbb{R})$ and $(u_1, u_2, \dots, u_n)(\pm\infty) = \mathbf{e}_{\pm}$, we may assume $\min_{x \in \mathbb{R}} q(x) = q(z)$. We denote respectively by z_2 and z_1 the first

points at which the solution $(u_1(x), u_2(x), \dots, u_n(x))$ intersects the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ when x moves from z towards ∞ and $-\infty$. For the case where $\theta \leq 0$, we integrate (2.4) with respect to x from z_1 to z and obtain

$$q'(z) - q'(z_1) + \theta(p(z) - p(z_1)) + \int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx \leq 0. \quad (2.20)$$

On the other hand we have:

- $q'(z) = 0$ because of $\min_{x \in \mathbb{R}} q(x) = q(z)$;
- $q(z_1) = \lambda_1$ follows from the fact that z_1 is on the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$. Since z_1 is the first point for $q(x)$ taking the value λ_1 when x moves from z to $-\infty$, we conclude that $q(z_1 + \delta) \leq \lambda_1$ for $z - z_1 > \delta > 0$ and $q'(z_1) \leq 0$;
- $p(z) < \eta_1$ since z is below the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$; $p(z_1) > \eta_1$ since z_1 is above the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$;
- let $\mathcal{F}_+ = \left\{ (u_1, u_2, \dots, u_n) \mid F(u_1, u_2, \dots, u_n) > 0, u_1, u_2, \dots, u_n \geq 0 \right\}$. Due to the fact that $(u_1(z_1), u_2(z_1), \dots, u_n(z_1))$ is on the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ and $(u_1(z), u_2(z), \dots, u_n(z)) \in \mathcal{Q}_{\lambda_2}$, $(u_1(z_1), u_2(z_1), \dots, u_n(z_1))$, we have $(u_1(z), u_2(z), \dots, u_n(z)) \in \mathcal{R}$ by (2.5). Because of **[H]**

and $F(u_1, u_2, \dots, u_n) = \sum_{i=1}^n \alpha_i u_i^{l_i} f_i(u_1, u_2, \dots, u_n)$, it is easy to see that

$$\left\{ (u_1(x), u_2(x), \dots, u_n(x)) \mid z_1 \leq x \leq z \right\} \subset \mathcal{R} \subset \mathcal{F}_+. \quad (2.21)$$

Therefore we have $\int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx > 0$.

Combining the above arguments, we obtain

$$q'(z) - q'(z_1) + \theta(p(z) - p(z_1)) + \int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx > 0, \quad (2.22)$$

which contradicts (2.20). Therefore when $\theta \leq 0$, $q(x) \geq \lambda_2$ for $x \in \mathbb{R}$. For the case where $\theta \geq 0$, integrating (2.4) with respect to x from z to z_2 yields

$$q'(z_2) - q'(z) + \theta(p(z_2) - p(z)) + \int_z^{z_2} F(u_1(x), u_2(x), \dots, u_n(x)) dx \leq 0. \quad (2.23)$$

In a similar manner, it can be shown that $q'(z_2) \geq 0$, $q'(z) = 0$, $p(z_2) > \eta$, $p(z) < \eta$, and

$$\int_z^{z_2} F(u_1(x), u_2(x), \dots, u_n(x)) dx > 0. \quad (2.24)$$

These together contradict (2.23). Consequently, (2.1) is proved and the proof is completed. \square

Remark 2.1 (N-barrier for lower bounds). When $\sigma_1 = \sigma_2 = c_{11} = c_{22} = 1$, $c_{12} = a_1$, and $c_{21} = a_2$ in (NDC-tw) with the asymptotic behavior $\mathbf{e}_- = (1, 0)$ and $\mathbf{e}_+ = (0, 1)$, we are led to the problem

$$\begin{cases} d_1(u^2)_{xx} + \theta u_x + u(1 - u - a_1 v) = 0, & x \in \mathbb{R}, \\ d_2(v^2)_{xx} + \theta v_x + v(1 - a_2 u - v) = 0, & x \in \mathbb{R}, \\ (u, v)(-\infty) = (1, 0), \quad (u, v)(+\infty) = (0, 1). \end{cases} \quad (2.25)$$

To satisfy the hypothesis [H], we let as in the proof of Corollary 1.3

$$\underline{u} = \min\left(1, \frac{1}{a_2}\right), \quad (2.26)$$

$$\underline{v} = \min\left(1, \frac{1}{a_1}\right). \quad (2.27)$$

For simplicity, we shall always assume the bistable condition $a_1, a_2 > 1$ for (2.25). This gives $\underline{u} = \frac{1}{a_2}$ and $\underline{v} = \frac{1}{a_1}$. We readily verify that under $a_1, a_2 > 1$, the quadratic curve

$$F(u, v) := \alpha u(1 - u - a_1 v) + \beta v(1 - a_2 u - v) = 0 \quad (2.28)$$

in the first quadrant of the uv -plane is a hyperbola for any $\alpha, \beta > 0$ and it passes through the equilibria $(0, 0)$, $(1, 0)$, $(1, 0)$ and $\left(\frac{a_1 - 1}{a_1 a_2 - 1}, \frac{a_2 - 1}{a_1 a_2 - 1}\right)$.

We are now in the position to follow the three steps in the proof of Proposition 1 to construct the N-barrier for the problem (2.25).

- (1) Since the line $\frac{u}{\underline{u}} + \frac{v}{\underline{v}} = 1$ is tangent to the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ at (u, v) in the first quadrant of the uv -plane, this leads to the following equations:

$$\frac{\alpha d_1 u}{\beta d_2 v} = \frac{\underline{v}}{\underline{u}}, \quad (2.29)$$

$$\frac{u}{\underline{u}} + \frac{v}{\underline{v}} = 1, \quad (2.30)$$

$$\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1. \quad (2.31)$$

By Lemma 5.1 (see Section 5), λ_1 is given by

$$\lambda_1 = \frac{1}{\frac{1}{\alpha d_1 u^2} + \frac{1}{\beta d_2 v^2}} = \frac{\alpha \beta d_1 d_2 u^2 v^2}{\alpha d_1 u^2 + \beta d_2 v^2}. \quad (2.32)$$

- (2) The u -coordinate of the u -intercept and the v -coordinate of the v -intercept of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ are $\sqrt{\frac{\lambda_1}{\alpha d_1}}$ and $\sqrt{\frac{\lambda_1}{\beta d_2}}$, respectively; the u -coordinate of the u -intercept and the v -coordinate of the line $\eta_1 = \alpha u + \beta v$ are $\frac{\eta_1}{\alpha}$ and $\frac{\eta_1}{\beta}$, respectively. Because of

$$\eta_1 = \sqrt{\lambda_1 \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right)}, \quad (2.33)$$

- when $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\alpha}{d_1}$, we clearly have

$$\frac{\eta_1}{\alpha} = \frac{1}{\alpha} \sqrt{\frac{\lambda_1 \alpha}{d_1}} = \sqrt{\frac{\lambda_1}{\alpha d_1}}, \quad (2.34)$$

$$\frac{\eta_1}{\beta} = \frac{1}{\beta} \sqrt{\frac{\lambda_1 \alpha}{d_1}} \leq \frac{\sqrt{\lambda_1}}{\beta} \sqrt{\frac{\beta}{d_2}} = \sqrt{\frac{\lambda_1}{\beta d_2}}; \quad (2.35)$$

- when $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\beta}{d_2}$, we clearly have

$$\frac{\eta_1}{\alpha} = \frac{1}{\alpha} \sqrt{\frac{\lambda_1 \beta}{d_2}} \leq \frac{\sqrt{\lambda_1}}{\alpha} \sqrt{\frac{\alpha}{d_1}} = \sqrt{\frac{\lambda_1}{\alpha d_1}}, \quad (2.36)$$

$$\frac{\eta_1}{\beta} = \frac{1}{\beta} \sqrt{\frac{\lambda_1 \beta}{d_2}} = \sqrt{\frac{\lambda_1}{\beta d_2}}. \quad (2.37)$$

This means that when $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\alpha}{d_1}$, the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ and the line $\eta_1 = \alpha u + \beta v$ possess the same u -coordinate of the u -intercept, i.e. $\sqrt{\frac{\lambda_1}{\alpha d_1}} = \frac{\eta_1}{\alpha}$; meanwhile, the inequality

$\frac{\eta_1}{\beta} \leq \sqrt{\frac{\lambda_1}{\beta d_2}}$ indicates that the v -coordinate of the v -intercept of the line $\eta_1 = \alpha u + \beta v$ is not larger than that of the v -intercept of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$. A similar conclusion can be drawn for the case of $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\beta}{d_2}$.

- (3) The fact that the line $\eta_1 = \alpha u + \beta v$ is tangent to the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ at (u, v) in the first quadrant of the uv -plane yields the following equations:

$$\frac{\alpha d_1 u}{\beta d_2 v} = \frac{\alpha}{\beta}, \quad (2.38)$$

$$\alpha u + \beta v = \eta_1, \quad (2.39)$$

$$\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2. \quad (2.40)$$

Employing Lemma 5.2 in Section 5, we obtain

$$\lambda_2 = \frac{\eta_1^2}{\frac{\alpha}{d_1} + \frac{\beta}{d_2}} = \frac{\eta_1^2 d_1 d_2}{\alpha d_2 + \beta d_1}. \quad (2.41)$$

The above three steps complete the construction of the N -barrier. Finally, we determine the line $\eta_2 = \alpha u + \beta v$ by setting

$$\eta_2 = \sqrt{\lambda_2 \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right)} \quad (2.42)$$

such that, as in step (ii), the line $\eta_2 = \alpha u + \beta v$ lies entirely below the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ in the first quadrant of the uv -plane. Combining (2.32), (2.33), (2.41) and (2.42), we arrive at

$$\begin{aligned} \eta_2 &= \sqrt{\lambda_2 \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right)} = \eta_1 \sqrt{\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) \frac{d_1 d_2}{\alpha d_2 + \beta d_1}} \\ &= \sqrt{\lambda_1 \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right)} \sqrt{\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) \frac{d_1 d_2}{\alpha d_2 + \beta d_1}} \\ &= \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) \sqrt{\frac{\alpha \beta d_1 d_2 u^2 v^2}{\alpha d_1 u^2 + \beta d_2 v^2} \frac{d_1 d_2}{\alpha d_2 + \beta d_1}} \\ &= d_1 d_2 u v \min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) \sqrt{\frac{\alpha \beta}{(\alpha d_1 u^2 + \beta d_2 v^2)(\alpha d_2 + \beta d_1)}} \\ &= \alpha \beta u v \min\left(\frac{d_1}{\alpha}, \frac{d_2}{\beta}\right) \sqrt{\frac{\alpha \beta}{(\alpha d_1 u^2 + \beta d_2 v^2)(\alpha d_2 + \beta d_1)}}. \end{aligned} \quad (2.43)$$

The lower bound η_2 coincides with that given in Corollary 1.3.

It follows immediately from step (ii) that there are two conditions: $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\alpha}{d_1}$ and $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\beta}{d_2}$. We show the N -barrier for each condition in Figure 1: the N -barrier for the case $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\alpha}{d_1}$ is shown in Figure 1(a), while the one for the case $\min\left(\frac{\alpha}{d_1}, \frac{\beta}{d_2}\right) = \frac{\beta}{d_2}$ is shown in Figure 1(b). We note that through the example of Figure 1 in which the N -barrier for the lower dimensional problem (2.25) is constructed, the N -barrier in the hyper-space in the proof of Proposition 1 become immediate.

Proposition 2 (Upper bound in NBMP). Suppose that $u_i(x) \in C^2(\mathbb{R})$ with $u_i(x) \geq 0$ ($i = 1, 2, \dots, n$) and satisfy the following differential inequalities and asymptotic behavior:

$$(BVP-I) \begin{cases} d_i (u_i^m)_{xx} + \theta(u_i)_x + u_i^{l_i} f_i(u_1, u_2, \dots, u_n) \geq 0, & x \in \mathbb{R}, \quad i = 1, 2, \dots, n, \\ (u_1, u_2, \dots, u_n)(-\infty) = \mathbf{e}_-, \quad (u_1, u_2, \dots, u_n)(\infty) = \mathbf{e}_+, \end{cases}$$

where \mathbf{e}_- and \mathbf{e}_+ are given by (1.3). If the hypothesis

[$\bar{\mathbf{H}}$] For $i = 1, 2, \dots, n$, there exist $\bar{u}_i > 0$ such that

$$f_i(u_1, u_2, \dots, u_n) \leq 0 \quad \text{whenever } (u_1, u_2, \dots, u_n) \in \bar{\mathcal{R}},$$

where $\bar{\mathcal{R}}$ is as defined in [\mathbf{H}]

holds, then we have for any $\alpha_i > 0$ ($i = 1, 2, \dots, n$)

$$\sum_{i=1}^n \alpha_i u_i(x) \leq \sqrt[m]{\left(\sum_{i=1}^n \frac{\alpha_i}{m-1\sqrt{d_i}}\right)^{2(m-1)} \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}}\right) \left(\max_{1 \leq i \leq n} \alpha_i d_i \bar{u}_i^m\right)} \quad (2.44)$$

Proof. We show by employing the N -barrier method as in the proof of Proposition 1 the upper bound given by (2.44). The construction of an appropriate N -barrier is the main ingredient of our proof. To do this, let

$$\mathcal{P}_\eta = \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \alpha_i u_i \geq \eta, \quad u_1, u_2, \dots, u_n \geq 0 \right\}; \quad (2.45)$$

$$\mathcal{Q}_\lambda = \left\{ (u_1, u_2, \dots, u_n) \mid \sum_{i=1}^n \alpha_i d_i u_i \geq \lambda, \quad u_1, u_2, \dots, u_n \geq 0 \right\}. \quad (2.46)$$

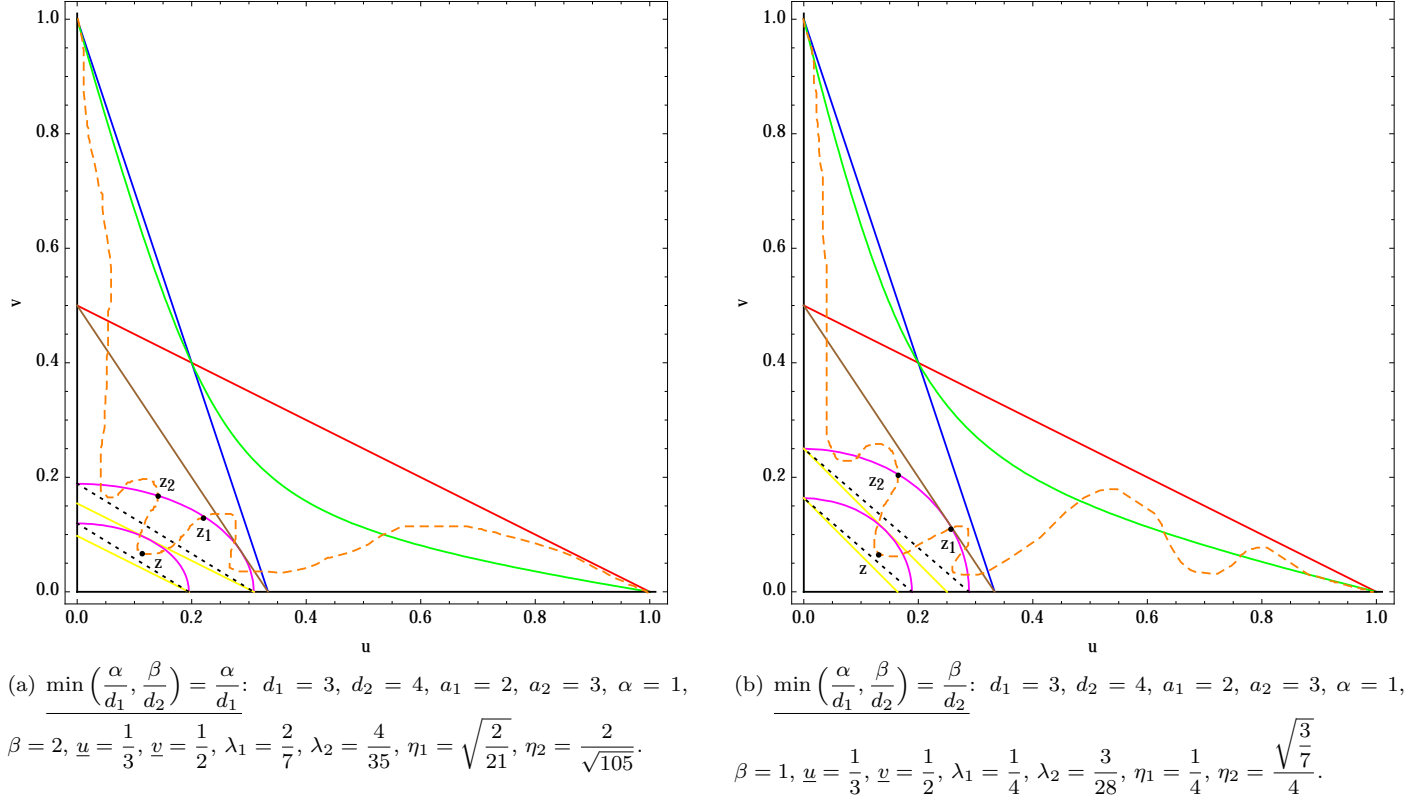


FIGURE 1. Red line: $1 - u - a_1 v = 0$; blue line: $1 - a_2 u - v = 0$; green curve: $F(u, v) := \alpha u(1 - u - a_1 v) + \beta v(1 - a_2 u - v) = 0$; brown line: $\frac{u}{\underline{u}} + \frac{v}{\underline{v}} = 1$, where \underline{u} and \underline{v} are given by (2.26) and (2.27); magenta ellipse (above): $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$, where λ_1 is given by (2.32); magenta ellipse (below): $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$, where λ_2 is given by (2.41); yellow line (above): $\alpha u + \beta v = \eta_1$, where η_1 is given by (2.33); yellow line (below): $\alpha u + \beta v = \eta_2$, where η_2 is given by (2.42); dashed orange curve: the solution $(u(x), v(x))$; dotted line (above): $\frac{u}{\sqrt{\frac{\lambda_1}{\alpha d_1}}} + \frac{v}{\sqrt{\frac{\lambda_1}{\beta d_2}}} = 1$; dotted line (below): $\frac{u}{\sqrt{\frac{\lambda_2}{\alpha d_1}}} + \frac{v}{\sqrt{\frac{\lambda_2}{\beta d_2}}} = 1$.

Recall (2.2) in the proof of Proposition 1. Adding the n equations in **(BVP-1)**, we obtain the equation

$$\frac{d^2 q(x)}{dx^2} + \theta \frac{dp(x)}{dx} + F(u_1(x), u_2(x), \dots, u_n(x)) \geq 0, \quad x \in \mathbb{R}, \quad (2.47)$$

where $F(u_1, u_2, \dots, u_n) := \sum_{i=1}^n \alpha_i u_i^{l_i} f_i(u_1, u_2, \dots, u_n)$.

We determine the positive parameters $\lambda_1, \lambda_2, \eta_1$ and η_2 such that the two hyper-ellipsoids $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$, $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$, and the two hyperplanes $\sum_{i=1}^n \alpha_i u_i = \eta_1$, $\sum_{i=1}^n \alpha_i u_i = \eta_2$ satisfy the relationship

$$\mathcal{P}_{\eta_2} \supset \mathcal{Q}_{\lambda_2} \supset \mathcal{P}_{\eta_1} \supset \mathcal{Q}_{\lambda_1} \supset \bar{\mathcal{R}}. \quad (2.48)$$

The hyper-ellipsoids $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$, $\sum_{i=1}^n \alpha_i d_i \bar{u}_i^m = \lambda_2$, and the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ form the N-barrier and it turns out that the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_2$ determines the upper bound in (2.44). We follow the three steps below to construct the N-barrier:

(1) Setting

$$\lambda_1 = \max_{1 \leq i \leq n} \alpha_i d_i \bar{u}_i^m, \quad (2.49)$$

the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ has the n intercepts $\left(\sqrt[m]{\frac{\lambda_1}{\alpha_1 d_1}}, 0, \dots, 0\right), \left(0, \sqrt[m]{\frac{\lambda_1}{\alpha_2 d_2}}, 0, \dots, 0\right), \dots$, and $\left(0, 0, \dots, 0, \sqrt[m]{\frac{\lambda_1}{\alpha_n d_n}}\right)$ and the hyperplane $\sum_{i=1}^n \frac{u_i}{\bar{u}_i} = 1$ has the n intercepts $(\bar{u}_1, 0, \dots, 0)$, $(0, \bar{u}_2, 0, \dots, 0), \dots$, and $(0, 0, \dots, 0, \bar{u}_n)$. It is easy to verify that $\mathcal{Q}_{\lambda_1} \supset \bar{\mathcal{R}}$ since $\bar{u}_j \leq \sqrt[m]{\frac{\lambda_1}{\alpha_j d_j}}$ for $j = 1, 2, \dots, n$. Indeed, we have

$$\begin{aligned} \left(\frac{\lambda_1}{\alpha_j d_j}\right)^{\frac{1}{m}} &= \left(\frac{\max_{1 \leq i \leq n} \alpha_i d_i \bar{u}_i^m}{\alpha_j d_j}\right)^{\frac{1}{m}} \\ &\geq \left(\frac{\alpha_j d_j \bar{u}_j^m}{\alpha_j d_j}\right)^{\frac{1}{m}} = \bar{u}_j. \end{aligned} \quad (2.50)$$

(2) Let the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ be tangent to the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ at (u_1, u_2, \dots, u_n) with $u_1, u_2, \dots, u_n > 0$ such that $\mathcal{P}_{\eta_1} \supset \mathcal{Q}_{\lambda_1}$. This leads to the following equations:

$$d_i u_i^{m-1} = d_j u_j^{m-1}, \quad i, j = 1, 2, \dots, n; \quad (2.51)$$

$$\sum_{i=1}^n \alpha_i u_i = \eta_1; \quad (2.52)$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1. \quad (2.53)$$

Employing Lemma 5.2 in Section 5, we obtain

$$\eta_1 = \lambda_1^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1} \sqrt[m]{d_i}} \right)^{\frac{m-1}{m}}. \quad (2.54)$$

(3) Setting

$$\lambda_2 = \eta_1^m \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}} \right), \quad (2.55)$$

the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$ has the n intercepts $\left(\sqrt[m]{\frac{\lambda_2}{\alpha_1 d_1}}, 0, \dots, 0\right), \left(0, \sqrt[m]{\frac{\lambda_2}{\alpha_2 d_2}}, 0, \dots, 0\right), \dots$, and $\left(0, 0, \dots, 0, \sqrt[m]{\frac{\lambda_2}{\alpha_n d_n}}\right)$ and the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$ has the n intercepts $\left(\frac{\eta_1}{\alpha_1}, 0, \dots, 0\right)$, $\left(0, \frac{\eta_1}{\alpha_2}, 0, \dots, 0\right), \dots$, and $\left(0, 0, \dots, 0, \frac{\eta_1}{\alpha_n}\right)$. It is easy to verify that $\mathcal{Q}_{\lambda_2} \supset \mathcal{P}_{\eta_1}$ since $\frac{\eta_1}{\alpha_j} \leq \left(\frac{\lambda_2}{\alpha_j d_j}\right)^{\frac{1}{m}}$

for $j = 1, 2, \dots, n$. Indeed, we have

$$\begin{aligned} \left(\frac{\lambda_2}{\alpha_j d_j} \right)^{\frac{1}{m}} &= \eta_1 \left(\frac{\max_{1 \leq i \leq n} d_i \alpha_i^{1-m}}{\alpha_j d_j} \right)^{\frac{1}{m}} \\ &\geq \eta_1 \left(\frac{d_j \alpha_j^{1-m}}{\alpha_j d_j} \right)^{\frac{1}{m}} = \frac{\eta_1}{\alpha_j}. \end{aligned} \quad (2.56)$$

Steps (i)~(iii) complete the construction of the N-barrier. As in step (ii), we determine η_2 by letting the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_2$ be tangent to the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2$ at (u_1, u_2, \dots, u_n) with $u_1, u_2, \dots, u_n > 0$ such that $\mathcal{P}_{\eta_2} \supset \mathcal{Q}_{\lambda_2}$. This leads to the following equations:

$$d_i u_i^{m-1} = d_j u_j^{m-1}, \quad i, j = 1, 2, \dots, n; \quad (2.57)$$

$$\sum_{i=1}^n \alpha_i u_i = \eta_2; \quad (2.58)$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_2. \quad (2.59)$$

Employing Lemma 5.2 in Section 5 again, we obtain

$$\eta_2 = \lambda_2^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1/\sqrt{d_i}}} \right)^{\frac{m-1}{m}}. \quad (2.60)$$

such that $\mathcal{P}_{\eta_2} \subset \mathcal{Q}_{\lambda_2}$. From (2.49), (2.54), (2.55) and (2.60), it follows immediately that η_2 is given by

$$\begin{aligned} \eta_2 &= \lambda_2^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1/\sqrt{d_i}}} \right)^{\frac{m-1}{m}} = \eta_1 \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}} \right)^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1/\sqrt{d_i}}} \right)^{\frac{m-1}{m}} \\ &= \lambda_1^{\frac{1}{m}} \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}} \right)^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1/\sqrt{d_i}}} \right)^{\frac{2(m-1)}{m}} \\ &= \left(\max_{1 \leq i \leq n} \alpha_i d_i \bar{u}_i^m \right)^{\frac{1}{m}} \left(\max_{1 \leq i \leq n} \frac{d_i}{\alpha_i^{m-1}} \right)^{\frac{1}{m}} \left(\sum_{i=1}^n \frac{\alpha_i}{m^{-1/\sqrt{d_i}}} \right)^{\frac{2(m-1)}{m}}. \end{aligned} \quad (2.61)$$

(An illustration of the N-barrier for $m = n = 2$ is given in Remark 2.2.)

As the proof of Proposition 1, we claim by contradiction that $q(x) \leq \lambda_2$ for $x \in \mathbb{R}$, from which (2.44) follows since the $\alpha_i > 0$ ($i = 1, 2, \dots, n$) are arbitrary and the relationship $\mathcal{P}_{\eta_2} \supset \mathcal{Q}_{\lambda_2}$ holds. Suppose that, contrary to our claim, there exists $z \in \mathbb{R}$ such that $q(z) > \lambda_2$. Since $u_i(x) \in C^2(\mathbb{R})$ and $(u_1, u_2, \dots, u_n)(\pm\infty) = \mathbf{e}_{\pm}$, we may assume $\max_{x \in \mathbb{R}} q(x) = q(z)$. We denote respectively by z_2 and z_1 the first points at which the solution

$(u_1(x), u_2(x), \dots, u_n(x))$ intersects the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ when x moves from z towards ∞ and $-\infty$. For the case where $\theta \leq 0$, we integrate (2.47) with respect to x from z_1 to z and obtain

$$q'(z) - q'(z_1) + \theta(p(z) - p(z_1)) + \int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx \geq 0. \quad (2.62)$$

On the other hand we have:

- $q'(z) = 0$ because of $\max_{x \in \mathbb{R}} q(x) = q(z)$;

- $q(z_1) = \lambda_1$ follows from the fact that z_1 is on the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$. Since z_1 is the first point for $q(x)$ taking the value λ_1 when x moves from z to $-\infty$, we conclude that $q(z_1 + \delta) \geq \lambda_1$ for $z - z_1 > \delta > 0$ and $q'(z_1) \geq 0$;
- $p(z) > \eta_1$ since z is above the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$; $p(z_1) < \eta_1$ since z_1 is below the hyperplane $\sum_{i=1}^n \alpha_i u_i = \eta_1$;
- let $\mathcal{F}_- = \left\{ (u_1, u_2, \dots, u_n) \mid F(u_1, u_2, \dots, u_n) < 0, u_1, u_2, \dots, u_n \geq 0 \right\}$. Due to the fact that $(u_1(z_1), u_2(z_1), \dots, u_n(z_1))$ is on the hyper-ellipsoid $\sum_{i=1}^n \alpha_i d_i u_i^m = \lambda_1$ and $(u_1(z), u_2(z), \dots, u_n(z)) \in \mathcal{Q}_{\lambda_2}$, $(u_1(z_1), u_2(z_1), \dots, u_n(z_1)), (u_1(z), u_2(z), \dots, u_n(z)) \in \mathcal{R}$ by (2.48). Because of $[\bar{\mathbf{H}}]$ and $F(u_1, u_2, \dots, u_n) = \sum_{i=1}^n \alpha_i u_i^{l_i} f_i(u_1, u_2, \dots, u_n)$, it is easy to see that

$$\left\{ (u_1(x), u_2(x), \dots, u_n(x)) \mid z_1 \leq x \leq z \right\} \subset \bar{\mathcal{R}} \subset \mathcal{F}_-. \quad (2.63)$$

Therefore we have $\int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx < 0$.

Combining the above arguments, we obtain

$$q'(z) - q'(z_1) + \theta(p(z) - p(z_1)) + \int_{z_1}^z F(u_1(x), u_2(x), \dots, u_n(x)) dx < 0, \quad (2.64)$$

which contradicts (2.62). Therefore when $\theta \leq 0$, $q(x) \leq \lambda_2$ for $x \in \mathbb{R}$. For the case where $\theta \geq 0$, integrating (2.47) with respect to x from z to z_2 yields

$$q'(z_2) - q'(z) + \theta(p(z_2) - p(z)) + \int_z^{z_2} F(u_1(x), u_2(x), \dots, u_n(x)) dx \geq 0. \quad (2.65)$$

In a similar manner, it can be shown that $q'(z_2) \leq 0$, $q'(z) = 0$, $p(z_2) < \eta$, $p(z) > \eta$, and

$$\int_z^{z_2} F(u_1(x), u_2(x), \dots, u_n(x)) dx < 0. \quad (2.66)$$

These together contradict (2.65). Consequently, (2.44) is proved and the proof is completed. \square

Remark 2.2 (N-barrier for upper bounds). We illustrate the construction of the N -barrier in Proposition 2 for the case when $m = n = 2$. For consistency, we use the setting in Remark 2.1.

(i) **Ellipse** $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ We first determine the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ by letting

$$\lambda_1 = \max(\alpha d_1 \bar{u}^2, \beta d_2 \bar{v}^2). \quad (2.67)$$

The u -coordinate of the u -intercept and the v -coordinate of the v -intercept of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ are $\sqrt{\frac{\lambda_1}{\alpha d_1}}$ and $\sqrt{\frac{\lambda_1}{\beta d_2}}$, respectively; the u -coordinate of the u -intercept and the v -coordinate of the line $\frac{u}{\bar{u}} + \frac{v}{\bar{v}} = 1$ are \bar{u} and \bar{v} , respectively. It turns out that

- when $\max(\alpha d_1 \bar{u}^2, \beta d_2 \bar{v}^2) = \alpha d_1 \bar{u}^2$, we have

$$\sqrt{\frac{\lambda_1}{\alpha d_1}} = \bar{u}, \quad \sqrt{\frac{\lambda_1}{\beta d_2}} = \bar{u} \sqrt{\frac{\alpha d_1}{\beta d_2}} \geq \bar{u} \sqrt{\frac{\bar{v}^2}{\bar{u}^2}} = \bar{v}; \quad (2.68)$$

– when $\max(\alpha d_1 \bar{u}^2, \beta d_2 \bar{v}^2) = \beta d_2 \bar{v}^2$, we have

$$\sqrt{\frac{\lambda_1}{\beta d_2}} = \bar{v}, \quad \sqrt{\frac{\lambda_1}{\alpha d_1}} = \bar{v} \sqrt{\frac{\beta d_2}{\alpha d_1}} \geq \bar{v} \sqrt{\frac{\bar{u}^2}{\bar{v}^2}} = \bar{u}. \quad (2.69)$$

This means that the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ lies entirely above the line $\frac{u}{\bar{u}} + \frac{v}{\bar{v}} = 1$ in the first quadrant of the uv -plane.

- (ii) **Line** $\eta_1 = \alpha u + \beta v$ Since the line $\eta_1 = \alpha u + \beta v$ is tangent to the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ at (u, v) in the first quadrant of the uv -plane, we have the following equations:

$$\frac{\alpha d_1 u}{\beta d_2 v} = \frac{\alpha}{\beta}, \quad (2.70)$$

$$\alpha u + \beta v = \eta_1, \quad (2.71)$$

$$\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1. \quad (2.72)$$

Employing Lemma 5.2 in Section 5, we obtain

$$\eta_1 = \sqrt{\lambda_1 \left(\frac{\alpha}{d_1} + \frac{\beta}{d_2} \right)}. \quad (2.73)$$

We note that the line $\eta_1 = \alpha u + \beta v$ lies entirely above the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ in the first quadrant of the uv -plane.

- (iii) **Ellipse** $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ We determine the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ by letting

$$\lambda_2 = \eta_1^2 \max \left(\frac{d_1}{\alpha}, \frac{d_2}{\beta} \right). \quad (2.74)$$

The u -coordinate of the u -intercept and the v -coordinate of the v -intercept of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ are $\sqrt{\frac{\lambda_2}{\alpha d_1}}$ and $\sqrt{\frac{\lambda_2}{\beta d_2}}$, respectively; the u -coordinate of the u -intercept and the v -coordinate of the line $\eta_1 = \alpha u + \beta v$ are $\frac{\eta_1}{\alpha}$ and $\frac{\eta_1}{\beta}$, respectively. It follows that

– when $\max \left(\frac{d_1}{\alpha}, \frac{d_2}{\beta} \right) = \frac{d_1}{\alpha}$, we have

$$\sqrt{\frac{\lambda_2}{\alpha d_1}} = \frac{\eta_1}{\alpha}, \quad \sqrt{\frac{\lambda_2}{\beta d_2}} = \eta_1 \sqrt{\frac{d_1}{\alpha \beta d_2}} \geq \eta_1 \sqrt{\frac{\alpha d_2}{\alpha \beta^2 d_2}} = \frac{\eta_1}{\beta}; \quad (2.75)$$

– when $\max \left(\frac{d_1}{\alpha}, \frac{d_2}{\beta} \right) = \frac{d_2}{\beta}$, we have

$$\sqrt{\frac{\lambda_2}{\beta d_2}} = \frac{\eta_1}{\beta}, \quad \sqrt{\frac{\lambda_2}{\alpha d_1}} = \eta_1 \sqrt{\frac{d_2}{\beta \alpha d_1}} \geq \eta_1 \sqrt{\frac{\beta d_1}{\alpha^2 \beta d_1}} = \frac{\eta_1}{\alpha}. \quad (2.76)$$

We see from the construction of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ that the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ lies entirely above the line $\eta_1 = \alpha u + \beta v$ in the first quadrant of the uv -plane.

The two ellipses $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1$ and $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$, and the line $\eta_1 = \alpha u + \beta v$ form the N -barrier. Finally, we find the tangent line of the ellipse $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2$ in the first quadrant of the uv -plane by determining the line $\eta_2 = \alpha u + \beta v$ as in step (ii):

$$\frac{\alpha d_1 u}{\beta d_2 v} = \frac{\alpha}{\beta}, \quad (2.77)$$

$$\alpha u + \beta v = \eta_2, \quad (2.78)$$

$$\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_2. \quad (2.79)$$

We obtain

$$\eta_2 = \sqrt{\lambda_2 \left(\frac{\alpha}{d_1} + \frac{\beta}{d_2} \right)} \quad (2.80)$$

or

$$\eta_2 = \left(\frac{\alpha}{d_1} + \frac{\beta}{d_2} \right) \sqrt{\max \left(\frac{d_1}{\alpha}, \frac{d_2}{\beta} \right) \max \left(\alpha d_1 \bar{u}^2, \beta d_2 \bar{v}^2 \right)} \quad (2.81)$$

by combining (2.67), (2.73), (2.74) and (2.80).

It is readily seen from that, depending on $\max(\alpha d_1 \bar{u}^2, \beta d_2 \bar{v}^2)$ and $\max \left(\frac{d_1}{\alpha}, \frac{d_2}{\beta} \right)$,

We are now in the position to prove Theorem 1.2.

Proof of Theorem 1.2. In Propositions 1 and 2, we obtain a lower and upper bound for $\sum_{i=1}^n \alpha_i u_i(x)$, respectively. Combining the results in Propositions 1 and 2, we immediately establish Theorem 1.2. \square

Remark 2.3. The tanh method [20, 8, 31, 32] allows us to find exact solutions to **(BVP)** with certain class of the nonlinearity. For instance, when $m = n = 2$, **(BVP)** with Zeldovich-type reaction terms ([42, ?, 41]) becomes

$$\begin{cases} d_1(u^2)_{xx} + \theta u_x + u^2(\sigma_1 - c_{11}u - c_{12}v) = 0, & x \in \mathbb{R}, \\ d_2(v^2)_{xx} + \theta v_x + v^2(\sigma_2 - c_{21}u - c_{22}v) = 0, & x \in \mathbb{R}, \\ (u, v)(-\infty) = \left(\frac{\sigma_1}{c_{11}}, 0 \right), \quad (u, v)(+\infty) = \left(0, \frac{\sigma_2}{c_{22}} \right). \end{cases} \quad (2.82)$$

Applying Theorem 5.3 (see Appendix 5.2), we see that when $c_{11} = 1$, $c_{22} = 2$, $d_1 = 3$, and $d_2 = 4$, (5.38) gives $\theta = 0$, $k_1 = 60$, $k_2 = 8$, $\sigma_1 = 240$, $\sigma_2 = 32$, $c_{12} = 27$, and $c_{21} = \frac{2}{5}$, and hence (2.82) admits the solution (see Figure 3)

$$\begin{cases} u(x) = 60(1 - \tanh x)^2, & x \in \mathbb{R}, \\ v(x) = 8(1 + \tanh x), & x \in \mathbb{R}. \end{cases} \quad (2.83)$$

Letting $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{3}$, it follows immediately that $\alpha u(x) + \beta v(x) = 30 \tanh^2 x - \frac{172}{3} \tanh x + \frac{98}{3}$ is monotonically decreasing in x . As a result,

$$\frac{16}{3} = \alpha u(\infty) + \beta v(\infty) \leq \alpha u(x) + \beta v(x) \leq \alpha u(-\infty) + \beta v(-\infty) = 120, \quad x \in \mathbb{R}. \quad (2.84)$$

On the other hand, upper and lower bounds given by Corollary 1.3 turn out to be

$$1.70 \approx \frac{80}{\sqrt{2211}} = \underline{\lambda} \leq \alpha u(x) + \beta v(x) \leq \bar{\lambda} = 180\sqrt{2} \approx 254.56, \quad x \in \mathbb{R}. \quad (2.85)$$

Thus, we verify Corollary 1.3 in this case. $\underline{u} = 80$, $\underline{v} = \frac{80}{9}$, $\bar{u} = 240$, $\bar{v} = 16$,

3. APPLICATION TO THE NONEXISTENCE OF THREE SPECIES TRAVELING WAVES: PROOF OF THEOREM 1.4

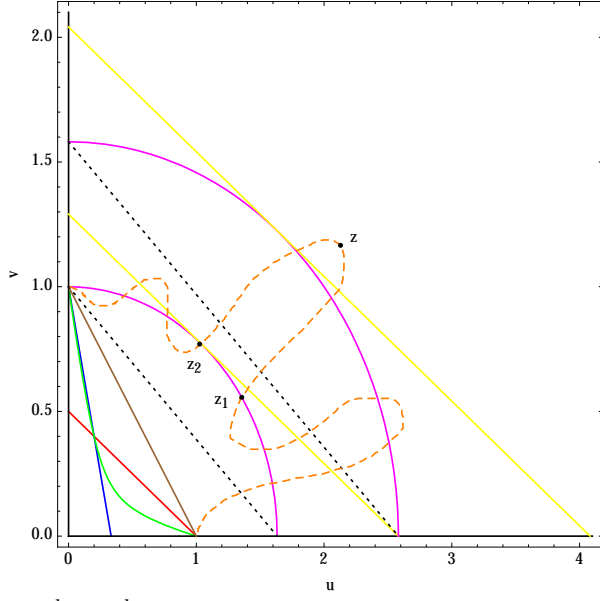
In this section, we prove Theorem 1.4 by contradiction.

Proof of Theorem 1.4. We first prove **(i)**. Suppose to the contrary that there exists a solution $(u(x), v(x), w(x))$ to the problem (1.23). Due to **[H1]**, we have $w_x(x_0) = 0$ and $w_{xx}(x_0) \leq 0$. Since $w(x)$ satisfies

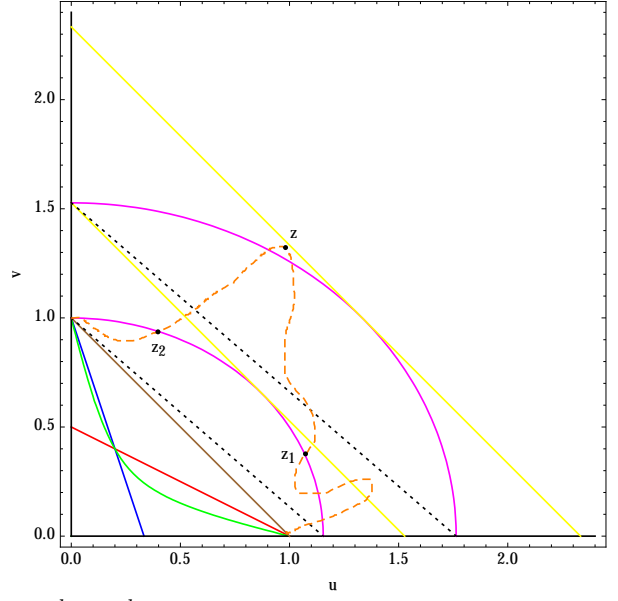
$$d_3(w^2)_{xx} + \theta w_x + w(\sigma_3 - c_{31}u - c_{32}v - c_{33}w) = 0 \quad (3.1)$$

and $(w^2)_{xx} = 2(w_x^2 + w w_{xx})$, we obtain

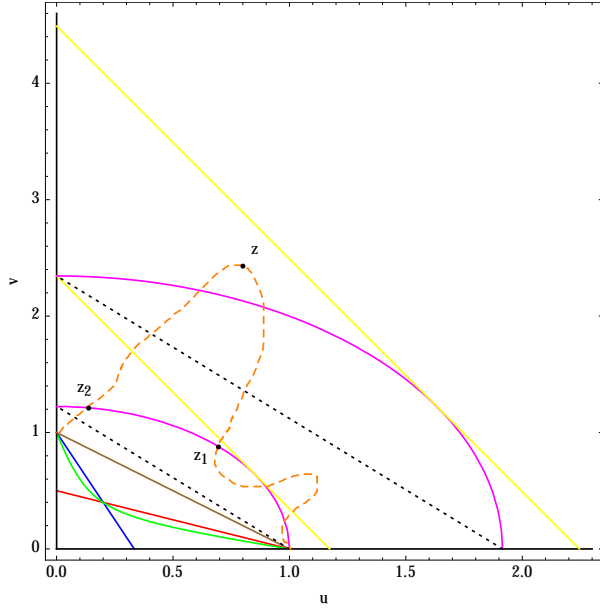
$$\sigma_3 - c_{31}u(x_0) - c_{32}v(x_0) - c_{33}w(x_0) \geq 0. \quad (3.2)$$



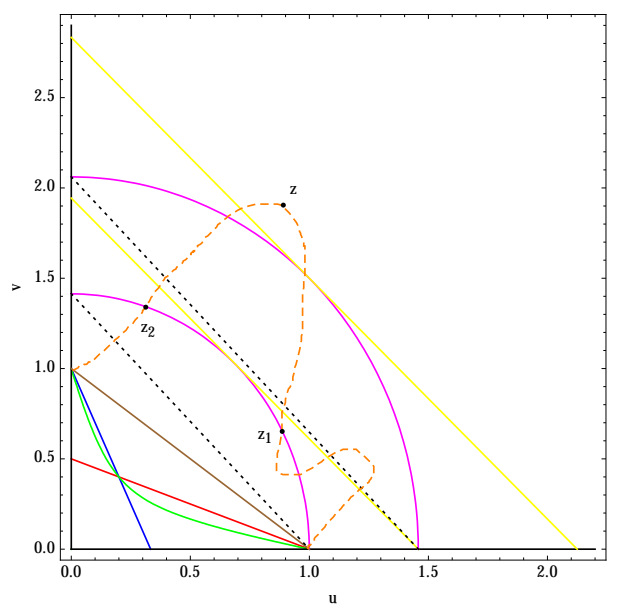
(a) $\frac{d_1}{\alpha} > \frac{d_2}{\beta}$, $\alpha d_1 \bar{u}^2 < \beta d_2 \bar{v}^2$: $d_2 = 4$, $\beta = 2$, $\lambda_1 = 8$, $\lambda_2 = 20$, $\eta_1 = 2\sqrt{\frac{5}{3}}$, $\eta_2 = 5\sqrt{\frac{2}{3}}$.



(b) $\frac{d_1}{\alpha} < \frac{d_2}{\beta}$, $\alpha d_1 \bar{u}^2 < \beta d_2 \bar{v}^2$: $d_2 = 4$, $\beta = 1$, $\lambda_1 = 4$, $\lambda_2 = \frac{28}{3}$, $\eta_1 = \sqrt{\frac{7}{3}}$, $\eta_2 = \frac{7}{3}$.



(c) $\frac{d_1}{\alpha} < \frac{d_2}{\beta}$, $\alpha d_1 \bar{u}^2 > \beta d_2 \bar{v}^2$: $d_2 = 4$, $\beta = \frac{1}{2}$, $\lambda_1 = 3$, $\lambda_2 = 11$, $\eta_1 = \frac{1}{2}\sqrt{\frac{11}{2}}$, $\eta_2 = \frac{11}{2\sqrt{6}}$.



(d) $\frac{d_1}{\alpha} > \frac{d_2}{\beta}$, $\alpha d_1 \bar{u}^2 > \beta d_2 \bar{v}^2$: $d_2 = 2$, $\beta = \frac{3}{4}$, $\lambda_1 = 3$, $\lambda_2 = \frac{51}{8}$, $\eta_1 = \frac{1}{2}\sqrt{\frac{17}{2}}$, $\eta_2 = \frac{17}{8}$.

FIGURE 2. Red line: $1 - u - a_1 v = 0$; blue line: $1 - a_2 u - v = 0$; green curve: $F(u, v) = 0$; brown line: $\frac{u}{\bar{u}} + \frac{v}{\bar{v}} = 1$, where \bar{u} and \bar{v} are given by (2.26) and (2.27); magenta ellipses: $\alpha d_1 u^2 + \beta d_2 v^2 = \lambda_1, \lambda_2$, where λ_1 (below) is given by (2.67) and λ_2 (above) by (2.74); yellow lines: $\alpha u + \beta v = \eta_1, \eta_2$, where η_1 (below) is given by (2.73) and η_2 (above) by (2.80); dashed orange curve: the solution $(u(x), v(x))$; dotted lines: $\sqrt{\alpha d_1} u + \sqrt{\beta d_2} v = \sqrt{\lambda_1}$ (below), $\sqrt{\lambda_2}$ (above); $\bar{u} = \bar{v} = 1$; $d_1 = 3$, $a_1 = 2$, $a_2 = 3$, $\alpha = 1$.

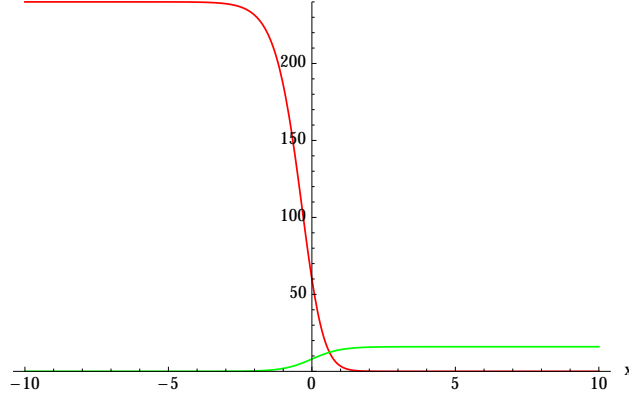


FIGURE 3. Red: $u(x) = 60(1 - \tanh x)^2$; green: $v(x) = 8(1 + \tanh x)$.

This lead to an upper bound of $w(x)$, i.e.

$$w(x) \leq w(x_0) \leq \frac{1}{c_{33}} (\sigma_3 - c_{31} u(x_0) - c_{32} v(x_0)) < \frac{\sigma_3}{c_{33}}, \quad x \in \mathbb{R}. \quad (3.3)$$

By virtue of the inequality $w(x) < \frac{\sigma_3}{c_{33}}$, the last two equations in (1.23) become

$$\begin{cases} d_1 (u^2)_{xx} + \theta u_x + u (\sigma_1 - c_{13} \sigma_3 c_{33}^{-1} - c_{11} u - c_{12} v) \leq 0, & x \in \mathbb{R}, \\ d_2 (v^2)_{xx} + \theta v_x + v (\sigma_2 - c_{23} \sigma_3 c_{33}^{-1} - c_{21} u - c_{22} v) \leq 0, & x \in \mathbb{R}. \end{cases} \quad (3.4)$$

By means of [H0] and [H2], we can employ Corollary 1.3 with $u_1 = u$, $u_2 = v$ and $\alpha_1 = c_{31}$, $\alpha_2 = c_{32}$ to obtain a lower bound of $c_{31} u(x) + c_{32} v(x)$, i.e.

$$c_{31} u(x) + c_{32} v(x) \geq d_1 d_2 \underline{u}_* \underline{v}_* \min \left(\frac{c_{31}}{d_1}, \frac{c_{32}}{d_2} \right) \sqrt{\frac{c_{31} c_{32}}{(c_{31} d_1 \underline{u}_*^2 + c_{32} d_2 \underline{v}_*^2) (c_{31} d_2 + c_{32} d_1)}}, \quad x \in \mathbb{R}. \quad (3.5)$$

However, [H3] yields

$$c_{31} u(x) + c_{32} v(x) \geq \lambda_* \geq \sigma_3, \quad x \in \mathbb{R}, \quad (3.6)$$

which contradicts (3.2). This completes the proof of (i). To prove (ii), an easy observation leads to

$$\begin{cases} d_1 (u^2)_{xx} + \theta u_x + u (\sigma_1 - c_{11} u - c_{12} v) > 0, & x \in \mathbb{R}, \\ d_2 (v^2)_{xx} + \theta v_x + v (\sigma_2 - c_{21} u - c_{22} v) > 0, & x \in \mathbb{R}, \end{cases} \quad (3.7)$$

since $w(x) > 0$, $x \in \mathbb{R}$. Letting $u_1 = u$, $u_2 = v$ and $\alpha_1 = c_{31}$, $\alpha_2 = c_{32}$, an upper bound of $c_{31} u(x) + c_{32} v(x)$ given by Corollary 1.3 is

$$c_{31} u(x) + c_{32} v(x) \leq \left(\frac{c_{31}}{d_1} + \frac{c_{32}}{d_2} \right) \sqrt{\max \left(\frac{d_1}{c_{31}}, \frac{d_2}{c_{32}} \right) \max \left(c_{31} d_1 \bar{u}^{*2}, c_{32} d_2 \bar{v}^{*2} \right)} := \lambda^*, \quad x \in \mathbb{R}, \quad (3.8)$$

where \bar{u}^* and \bar{v}^* are defined in [H5]. It follows from the last inequality that

$$\begin{aligned} 0 &= d_3 (w^2)_{xx} + \theta w_x + w (\sigma_3 - c_{31} u - c_{32} v - c_{33} w) \\ &\geq d_3 (w^2)_{xx} + \theta w_x + w (\sigma_3 - \lambda^* - c_{33} w). \end{aligned} \quad (3.9)$$

On the other hand, [H4] leads to the fact that $w_x(x_0) = 0$ and $w_{xx}(x_0) \geq 0$, and hence

$$\sigma_3 - \lambda^* - c_{33} w(x_0) \leq 0. \quad (3.10)$$

or

$$w(x) \geq w(x_0) \geq \frac{1}{c_{33}} (\sigma_3 - \lambda^*), \quad x \in \mathbb{R}. \quad (3.11)$$

However, this is a contradiction with [H6]. We complete the proof of (ii).

□

4. CONCLUDING REMARKS

In this paper, we have shown the NBMP for **(BVP)** with $m > 1$, and apply it to establish the nonexistence of three species waves in (1.23) under certain conditions. In particular, the upper and lower bounds given by the NBMP are verified by using exact solutions.

The N-barrier method is still under investigation, and there is a number of open problems concerning NBMP. We point out some of them for further study:

- *NBMP for periodic solutions:* As we can see from [13], **(NDC)** admits periodic stationary solutions under certain conditions on the parameters. Motivated by this work, we show in Theorem 5.4 (see Section 5.3) that for the three-species case (1.23) also admits periodic solutions under certain conditions on the parameters. The question is how to correct the N-barrier method adapted for periodic solutions?
- *NBMP for multi-dimensional equations:* The N-barrier method has not yet been applied to multi-dimensional equations since there is still a lack of systematic formulation of the method in the multi-dimensional case. The difficulty is to construct appropriate N-barriers corresponding to operator like Δu , ∇u , $\Delta(u^2)$ etc..
- *NBMP for strongly-coupled equations:* The N-barrier method developed to study (1.1) can also be applied to a wide class of elliptic systems, for instance, the system **(SKT-tw)** in which diffusion, self-diffusion, and cross-diffusion are strongly coupled.

These are left as the future work.

5. APPENDIX

5.1. Algebraic solutions.

Lemma 5.1. *For $\Theta, \Lambda > 0$, if*

$$\alpha_i d_i u_i^{m-1} = \alpha_j d_j u_j^{m-1}, \quad i, j = 1, 2, \dots, n; \quad (5.1)$$

$$\sum_{i=1}^n \frac{u_i}{u_i} = \Theta; \quad (5.2)$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \Lambda, \quad (5.3)$$

we have

$$\Lambda = \Theta^m \left(\sum_{i=1}^n \frac{1}{\alpha_i d_i u_i^m} \right)^{1-m}. \quad (5.4)$$

Proof. Due to (5.1), we may assume

$$u_i = \sqrt[m-1]{\frac{\prod_{j=1}^n \alpha_j d_j u_j}{\alpha_i d_i u_i}} K, \quad i = 1, 2, \dots, n \quad (5.5)$$

for some $K > 0$. It follows immediately from (5.2) that K is determined by

$$K = \frac{\Theta}{\sum_{i=1}^n \left(\frac{1}{u_i} \sqrt[m-1]{\frac{\prod_{j=1}^n \alpha_j d_j u_j}{\alpha_i d_i u_i}} \right)}, \quad (5.6)$$

and hence

$$\begin{aligned}
u_i &= \sqrt[m-1]{\frac{\prod_{j=1}^n \alpha_j d_j \underline{u}_j}{\alpha_i d_i \underline{u}_i}} \frac{\Theta}{\sum_{i=1}^n \left(\frac{1}{\underline{u}_i} \sqrt[m-1]{\frac{\prod_{j=1}^n \alpha_j d_j \underline{u}_j}{\alpha_i d_i \underline{u}_i}} \right)} \\
&= \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i}} \frac{\Theta}{\sum_{i=1}^n \left(\frac{1}{\underline{u}_i} \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i}} \right)} \\
&= \frac{\Theta}{\sum_{i=1}^n \left(\frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)} \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i}}.
\end{aligned} \tag{5.7}$$

Therefore, Λ is given by

$$\begin{aligned}
\Lambda &= \sum_{i=1}^n \alpha_i d_i u_i^m = \sum_{i=1}^n \alpha_i d_i \left(\frac{\Theta}{\sum_{i=1}^n \left(\frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)} \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i}} \right)^m \\
&= \Theta^m \left(\sum_{i=1}^n \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)^{-m} \left(\sum_{i=1}^n \frac{\alpha_i d_i}{\sqrt[m-1]{(\alpha_i d_i \underline{u}_i)^m}} \right) \\
&= \Theta^m \left(\sum_{i=1}^n \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)^{-m} \left(\sum_{i=1}^n \frac{\sqrt[m-1]{(\alpha_i d_i)^{m-1}}}{\sqrt[m-1]{(\alpha_i d_i)^m (\underline{u}_i)^m}} \right) \\
&= \Theta^m \left(\sum_{i=1}^n \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)^{-m} \left(\sum_{i=1}^n \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right) = \Theta^m \left(\sum_{i=1}^n \frac{1}{\sqrt[m-1]{\alpha_i d_i \underline{u}_i^m}} \right)^{1-m}.
\end{aligned}$$

□

Lemma 5.2. For $\Theta, \Lambda > 0$, if

$$d_i u_i^{m-1} = d_j u_j^{m-1}, \quad i, j = 1, 2, \dots, n; \tag{5.8}$$

$$\sum_{i=1}^n \alpha_i u_i = \Theta; \tag{5.9}$$

$$\sum_{i=1}^n \alpha_i d_i u_i^m = \Lambda, \tag{5.10}$$

we have

$$\Lambda = \Theta^m \left(\sum_{i=1}^n \frac{\alpha_i}{\sqrt[m-1]{d_i}} \right)^{1-m}. \tag{5.11}$$

Proof. Lemma 5.2 follows from letting $\underline{u}_i = \frac{1}{\alpha_i}$ in Lemma 5.1.

□

5.2. Exact solutions using Tanh method. Enlightened by the works of [20, 8, 31, 32], our idea is to look for a monotone solution with a hyperbolic tangent profile. We make the following ansatz for solving (2.82):

$$\begin{cases} u(x) = k_1 (1 - \tanh x)^2, & x \in \mathbb{R}, \\ v(x) = k_2 (1 + \tanh x), & x \in \mathbb{R}, \end{cases} \quad (5.12)$$

where k_1 and k_2 are positive constants to be determined. Since the derivative of $\tanh x$ is expressible in terms of itself, i.e. $\frac{d}{dx} \tanh x = 1 - \tanh^2 x$, we see that the n th derivative of a polynomial in $\tanh x$ with any order is also a polynomial in $\tanh x$. Inserting ansatz (5.12) into (2.82), this fact enables us to get

$$\begin{aligned} d_1(u^2)_{xx} + \theta u_x + u(\sigma_1 - c_{11}u - c_{12}v) &= u(\zeta_0 + \zeta_1 T(x) + \zeta_2 T^2(x) + \zeta_3 T^3(x)), \\ d_2(v^2)_{xx} + \theta v_x + v(\sigma_2 - c_{21}u - c_{22}v) &= v(\xi_0 + \xi_1 T(x) + \xi_2 T^2(x) + \xi_3 T^3(x)), \end{aligned}$$

where $T(x) := \tanh x$,

$$\zeta_0 = -c_{11}k_1^2 - c_{12}k_1k_2 + 12d_1k_1 - 2\theta + \sigma_1k_1, \quad (5.13)$$

$$\zeta_1 = 4c_{11}k_1^2 + c_{12}k_1k_2 + 8d_1k_1 - 2\theta - 2\sigma_1k_1, \quad (5.14)$$

$$\zeta_2 = -6c_{11}k_1^2 + c_{12}k_1k_2 - 32d_1k_1 + \sigma_1k_1, \quad (5.15)$$

$$\zeta_3 = 4c_{11}k_1^2 - c_{12}k_1k_2 - 8d_1k_1, \quad (5.16)$$

$$\zeta_4 = 20d_1k_1 - c_{11}k_1^2, \quad (5.17)$$

and

$$\xi_0 = -c_{22}k_2^2 - c_{21}k_1k_2 + 2d_2k_2 + \theta + \sigma_2k_2, \quad (5.18)$$

$$\xi_1 = -2c_{22}k_2^2 + c_{21}k_1k_2 - 6d_2k_2 - \theta + \sigma_2k_2, \quad (5.19)$$

$$\xi_2 = -c_{22}k_2^2 + c_{21}k_1k_2 - 2d_2k_2, \quad (5.20)$$

$$\xi_3 = 6d_2k_2 - c_{21}k_1k_2. \quad (5.21)$$

Equating the coefficients of powers of $T(x)$ to zero yields a system of 9 equations:

$$\zeta_i = 0 \quad (i = 0, 1, 2, 3, 4), \quad \xi_i = 0 \quad (i = 0, 1, 2, 3). \quad (5.22)$$

It turns out that, with d_1 , d_2 , c_{11} , and c_{22} being free parameters, (5.22) can be solved to give

$$\begin{aligned} k_1 &= \frac{20d_1}{c_{11}}, & \sigma_1 &= 80d_1, & c_{12} &= \frac{18c_{22}d_1}{d_2}, & \theta &= 0, \\ k_2 &= \frac{4d_2}{c_{22}}, & \sigma_2 &= 8d_2, & c_{21} &= \frac{3c_{11}d_2}{10d_1}. \end{aligned} \quad (5.23)$$

The result obtained is summarized in the following

Theorem 5.3. *System (2.82) has a solution of the form (5.12) provided that (5.38) holds.*

5.3. Exact solutions of (SKT-tw). Inspired by the exact periodic solutions proposed in [13], we make the ansatz for solving (1.23) as follows:

$$\begin{cases} u(x) = k_1 + m_1 \cos(\mu x), & x \in \mathbb{R}, \\ v(x) = k_2 + m_2 \cos(\mu x), & x \in \mathbb{R}, \\ w(x) = k_3 + m_3 \cos(\mu x), & x \in \mathbb{R}, \end{cases} \quad (5.24)$$

where $\mu \neq 0$, $k_1, k_2, k_3 > 0$ and $m_1 \neq 0, m_2 \neq 0, m_3 \neq 0$ with $|m_1| \leq k_1$, $|m_2| \leq k_2$, and $|m_3| \leq k_3$ are constants to be determined. Inserting ansatz (5.24) into (1.23), we obtain

$$\begin{aligned} d_1(u^2)_{xx} + \theta u_x + u(\sigma_1 - c_{11}u - c_{12}v - c_{13}w) &= \zeta_0 + \zeta_1 \mathcal{C}(x) + \zeta_2 \mathcal{C}^2(x) + \zeta_3 \mathcal{S}(x), \\ d_2(v^2)_{xx} + \theta v_x + v(\sigma_2 - c_{21}u - c_{22}v - c_{23}w) &= \xi_0 + \xi_1 \mathcal{C}(x) + \xi_2 \mathcal{C}^2(x) + \xi_3 \mathcal{S}(x), \\ d_3(w^2)_{xx} + \theta w_x + w(\sigma_3 - c_{31}u - c_{32}v - c_{33}w) &= \varsigma_0 + \varsigma_1 \mathcal{C}(x) + \varsigma_2 \mathcal{C}^2(x) + \varsigma_3 \mathcal{S}(x), \end{aligned}$$

where $\mathcal{C}(x) := \cos(\mu x)$, $\mathcal{S}(x) := \sin(\mu x)$ and

$$\zeta_0 = -c_{11} k_1^2 - c_{12} k_2 k_1 - c_{13} k_3 k_1 + 2 d_1 \mu^2 m_1^2 + k_1 \sigma_1, \quad (5.25)$$

$$\begin{aligned} \zeta_1 = & -2 c_{11} k_1 m_1 - c_{12} k_2 m_1 - c_{13} k_3 m_1 - c_{12} k_1 m_2 \\ & - c_{13} k_1 m_3 - 2 d_1 k_1 \mu^2 m_1 + m_1 \sigma_1, \end{aligned} \quad (5.26)$$

$$\zeta_2 = -c_{11} m_1^2 - c_{12} m_2 m_1 - c_{13} m_3 m_1 - 4 d_1 \mu^2 m_1^2, \quad (5.27)$$

$$\zeta_3 = \theta \mu m_1, \quad (5.28)$$

$$\xi_0 = -c_{22} k_2^2 - c_{21} k_1 k_2 - c_{23} k_3 k_2 + 2 d_2 \mu^2 m_2^2 + k_2 \sigma_2, \quad (5.29)$$

$$\begin{aligned} \xi_1 = & -c_{21} k_2 m_1 - c_{21} k_1 m_2 - 2 c_{22} k_2 m_2 - c_{23} k_3 m_2 \\ & - c_{23} k_2 m_3 - 2 d_2 k_2 \mu^2 m_2 + m_2 \sigma_2, \end{aligned} \quad (5.30)$$

$$\xi_2 = -c_{22} m_2^2 - c_{21} m_1 m_2 - c_{23} m_3 m_2 - 4 d_2 \mu^2 m_2^2, \quad (5.31)$$

$$\xi_3 = \theta \mu m_2, \quad (5.32)$$

$$\varsigma_0 = -c_{33} k_3^2 - c_{31} k_1 k_3 - c_{32} k_2 k_3 + 2 d_3 \mu^2 m_3^2 + k_3 \sigma_3, \quad (5.33)$$

$$\begin{aligned} \varsigma_1 = & -c_{31} k_3 m_1 - c_{32} k_3 m_2 - c_{31} k_1 m_3 - c_{32} k_2 m_3 \\ & - 2 c_{33} k_3 m_3 - 2 d_3 k_3 \mu^2 m_3 + m_3 \sigma_3, \end{aligned} \quad (5.34)$$

$$\varsigma_2 = -c_{33} m_3^2 - c_{31} m_1 m_3 - c_{32} m_2 m_3 - 4 d_3 \mu^2 m_3^2, \quad (5.35)$$

$$\varsigma_3 = \theta \mu m_3. \quad (5.36)$$

Equating the coefficients of powers of $\mathcal{C}(x)$ and $\mathcal{S}(x)$ to zero yields a system of 12 equations:

$$\zeta_i = 0 \quad (i = 0, 1, 2, 3), \quad \xi_i = 0 \quad (i = 0, 1, 2, 3), \quad \varsigma_i = 0 \quad (i = 0, 1, 2, 3). \quad (5.37)$$

It turns out that, with m_i , d_i , c_{ij} ($i, j = 1, 2, 3, i \neq j$), and μ being free parameters, (5.37) can be solved to give

$$k_1 = -m_1, \quad \sigma_1 = 2 (c_{12} m_2 + c_{13} m_3 + 3 d_1 \mu^2 m_1), \quad c_{11} = -m_1^{-1} (c_{12} m_2 + c_{13} m_3 + 4 d_1 \mu^2 m_1), \quad (5.38)$$

$$k_2 = m_2, \quad \sigma_2 = -2 (c_{21} m_1 + 3 d_2 \mu^2 m_2), \quad c_{22} = -m_2^{-1} (c_{21} m_1 + c_{23} m_3 + 4 d_2 \mu^2 m_2),$$

$$k_3 = m_3, \quad \sigma_3 = -2 (c_{31} m_1 + 3 d_3 \mu^2 m_3), \quad c_{33} = -m_3^{-1} (c_{31} m_1 + c_{32} m_2 + 4 d_3 \mu^2 m_3),$$

$$\theta = 0.$$

We note that $\zeta_3 = \xi_3 = \varsigma_3 = 0$ immediately leads to $\theta = 0$. The result obtained is summarized in the following

Theorem 5.4. *System (1.23) has a solution of the form (5.24) provided that (5.38) holds.*

In view of Theorem 5.4, (1.23) has the solution

$$\begin{cases} u(x) = \frac{1}{10} (1 - \cos(2x)), & x \in \mathbb{R}, \\ v(x) = \frac{1}{11} (1 + \cos(2x)), & x \in \mathbb{R}, \\ w(x) = \frac{1}{12} (1 + \cos(2x)), & x \in \mathbb{R}, \end{cases} \quad (5.39)$$

when $d_i = \sigma_i = c_{ii} = 1$ ($i = 1, 2, 3$), $c_{12} = \frac{1067}{60}$, $c_{13} = 1$, $c_{21} = \frac{175}{11}$, $c_{23} = \frac{6}{11}$, $c_{31} = 15$, $c_{32} = \frac{11}{12}$, and $\theta = 0$. The resulting profiles of (5.39) are shown in Figure 4.

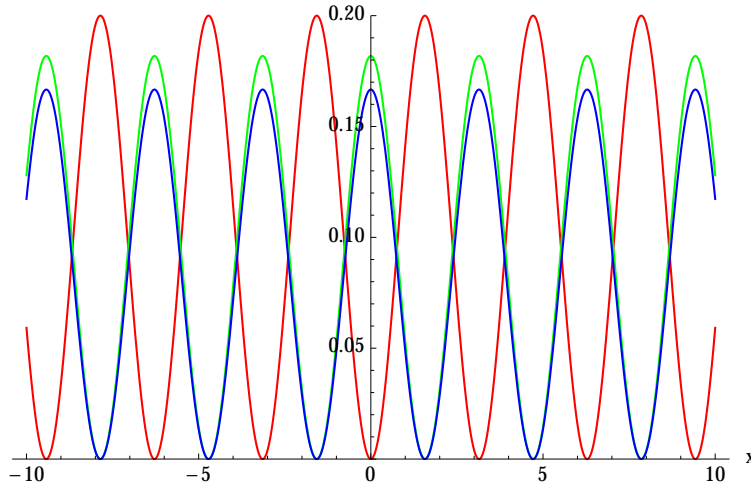


FIGURE 4. Red: $u(x) = \frac{1}{10}(1 - \cos(2x))$; green: $v(x) = \frac{1}{11}(1 + \cos(2x))$; blue: $w(x) = \frac{1}{12}(1 + \cos(2x))$.

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